RESEARCH ON NUMBER THEORY AND SMARANDACHE NOTIONS

PROCEEDINGS OF THE FIFTH INTERNATIONAL CONFERENCE ON NUMBER THEORY AND SMARANDACHE NOTIONS

Edited by
ZHANG WENPENG
Department of Mathematics
Northwest University
Xi'an, P.R.China

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Preface

This Book is devoted to the proceedings of the fifth International Conference on Number Theory and Smarandache Notions held in Shangluo during March 27-30, 2009. The organizers were myself and Professor Chao Li from Shangluo University. The conference was supported by Shangluo University and there were more than 100 participants. We had three foreign guests, Professor K.Chakraborty from India, Professor Imre Katai from Hungary, Professor S. Kanemitsu from Japan. The conference was a great success and will give a strong impact on the development of number theory in general and Smarandache Notions in particular. We hope this will become a tradition in our country and will continue to grow. And indeed we are planning to organize the sixth conference in coming March which will be held in Tianshui, a beautiful city of Gansu .

In the volume we assemble not only those papers which were presented at the conference but also those papers which were submitted later and are concerned with the Smarandache type problems or other mathematical problems.

There are a few papers which are not directly related to but should fall within the scope of Smarandache type problems. They are 1. Y. Wang, Smarandache sequence of ulam numbers; 2. H. Gunarto and A. A. K. Majumdar, On numerical values of Z(n); 3. K. Nagarajan, A. Nagarajan and S. Somasundaram, M-graphoidal path covers of a graph; etc.

Other papers are concerned with the number-theoretic Smarandache problems and will enrich the already rich stock of results on them.

Readers can learn various techniques used in number theory and will get familiar with the beautiful identities and sharp asymptotic formulas obtained in the volume.

Researchers can download books on the Smarandache notions from the following open source Digital Library of Science:

www.gallup.unm.edu/~smarandache/eBooks-otherformats.htm.

Wenpeng Zhang

Contents

The Fifth International Conference on Number Theory

and Smarandache Notions

The opening ceremony of the conference is occurred in Shangluo University (http://www.slxy.cn), which is attracting more and more people from other countries to study Chinese Calligraphy, Chinese Drawing and Chinese Culture.

Professor Chao Li

Professor Nianliang Wang

Professor Shigeru Kanemitsu

Professor Wenpeng Zhang

Professor Yongzhuang Chen


第五届国际数论与Smarandache问题研讨会合影留念
陕西 商洛 2009.3

An equation related to the Smarandache power function

Jianping Wang

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Abstract For any positive integer x, let $\varphi(x)$ and $f(x)$ denote the Euler totient function and the Smarandache power function of x respectively. Let n be a fixed positive integer. Using elementary method, we prove that the equation $f(x^n) = \varphi(x)$ has at least two positive integer solutions x, but it has only finitely many positive integer solutions.

Keywords Euler totient function, Smarandache power function, equation, solution.

§1. Introduction and Results

Let N^+ be the set of all positive integers. For any positive integer x, let $P(x)$ denotes the set of distinct prime divisors of x, and let

$$f(x) = \min\left\{m \mid m \in N^+,\ x \mid m^m,\quad P(m) = P(x)\right\}. \tag{1}$$

Then $f(x)$ is called the Smarandache power function of x. Many properties of this arithmetical function are discussed, see references [1], [2], [3] and [4].

Let n be a positive integer. Let $\varphi(x)$ denotes the Euler totient function of x. In this paper, we consider the equation

$$f(x^n) = \varphi(x),\ x \in N^+. \tag{2}$$

Recently, Chengliang Tian and Xiaoyan Li [4] gave all positive integer solutions of equation (2) for $n = 1,\ 2$ and 3. Simultaneously, they proposed the following:

Conjecture. For any fixed positive integer n, the equation (2) has only finitely many positive integer solutions x.

In this paper, we use the elementary method to prove the following general results.

Theorem 1. For any fixed positive integer n, let

$$r = \min\left\{m \mid m \in N^+,\quad n \leq 2^{m-1}\left(1 - \frac{1}{m}\right)\right\}. \tag{3}$$

Then $x = 2^r$ is a solution of equation (2).

Since $x = 1$ is a solution of (2), by Theorem 1, we get the following Corollary immediately.

Corollary 1. For any fixed positive integer n, the equation (2) has at least two positive integer solutions.

On the other hand, we can also obtain an explicit upper bound estimate for the solutions of equation (2) as follows:

Theorem 2. All positive integer solutions x of equation (2) satisfy $x < 256n^2$.

From Theorem 2, we may immediately deduce that the above Conjecture is correct. That is, we have the following:

Corollary 2. For any fixed positive integer n, the equation (2) has only finite many positive integer solutions.

§2. Proof of the theorem 1

For any positive integer m, let

$$g(m) = 2^{m-1}\left(1 - \frac{1}{m}\right), \quad h(m) = 2^{m-2}\left(1 - \frac{2}{m}\right). \tag{4}$$

Lemma 1. If $m > 1$, then $h(m+1) < g(m) < h(m+2)$.

Proof. By (4), if $h(m+1) \geq g(m)$, then $1 - \dfrac{2}{m+1} \geq 1 - \dfrac{1}{m}$. But, since $m \geq 2$, it is impossible. So we have $h(m+1) < g(m)$. Similarly, we can prove that $g(m) < h(m+2)$. Lemma 1 is proved.

Lemma 2. Let s be a positive integer. Then $x = 2^s$ is a solution of (2) if and only if s satisfies

$$h(s) < n \leq g(s). \tag{5}$$

Proof. Let $x = 2^s$. Then we have $\varphi(x) = 2^{s-1}$. By (4), if $n \leq g(s)$, then $sn \leq 2^{s-1}(s-1)$, whence we get

$$x^n \mid \varphi(x)^{\varphi(x)}. \tag{6}$$

On the other hand, if $h(s) < n$, then $2^{s-2}(s-2) < sn$. It implies that

$$x^n \nmid m^m, \quad m = 2^t, \quad t = 0,\ 1,\ \cdots,\ s-2. \tag{7}$$

Therefore, by (1), we see from (6) and (7) that $f(x^n) = \varphi(x)$ for $x = 2^s$, namely, it is a positive integer solution of equation (2).

By the same method, we can prove that if $x = 2^s$ is a solution of (2), then s satisfies (5). This proves Lemma 1.

Proof of Theorem 1. We see from (3) and (4) that $g(r-1) < n \leq g(r)$. If $n \leq h(r)$, then we have $g(r-1) < h(r)$. However, since $r \geq 2$, by Lemma 1, it is impossible. So we have $h(r) < n \leq g(r)$. Thus, by Lemma 2, $x = 2^r$ is a solution of equation (2). This completes the proof of Theorem 1.

§2. Proof of the theorem 2

Lemma 3. For any fixed positive integer n, the equation (2) has at most two positive integer solutions with the form

$$x = 2^s, \quad s \in N^+. \tag{8}$$

Moreover, if (2) has exactly two solutions with the form (8), then $x = 2^r$ and 2^{r+1}.

Proof. We now assume that equation (2) has three solutions $x = 2^{s_i} \quad (i = 1,\ 2,\ 3)$ with $s_1 < s_2 < s_3$. Then, by Lemma 2, we have $n \leq g(s_i)$ $(i = 1,\ 2,\ 3)$. Further, by (3), we get $r \leq s_1$ and $r+2 \leq s_3$. Furthermore, using Lemma 2 again, we have $h(r+2) \leq h(s_3) < n \leq g(r)$. However, by Lemma 1, it is impossible. Thus, equation (2) has at most two positive integer solutions with form (8).

In addition, by Theorem 1, using the same method, we can prove the second half of Lemma 3. This proves Lemma 3.

Lemma 4. If $x > 1$, then $\varphi(x) > \dfrac{x}{4\log x}$.

Proof. This is Lemma 2 of [5].

Proof of Theorem 2. By the result of [4], the theorem 2 holds for $n \leq 3$. So we can assume that $n > 3$. Since $\varphi(1) = \varphi(2) = 1$, the equation (2) has only the positive integer solution $x = 1$ with $x \leq 2$. Let x be a solution of equation (2) with $x > 2$. Since $\varphi(x)$ is even if $x > 2$, then x must be even. By Lemma 2, if $x = 2^s$, then $h(s) < n \leq g(s)$. Hence, by (4), we get that $s > 3$ and

$$h(s) = 2^{s-2}\left(1 - \frac{2}{s}\right) = \frac{x}{4}\left(1 - \frac{2}{s}\right) < n, \tag{9}$$

whence we obtain $x < 12n$.

If $x \neq 2^s$, then x has the factorization

$$x = 2^{r_0} p_1^{r_1} \cdots p_k^{r_k}, \tag{10}$$

where $p_i \ \ (i = 1, \cdots, k)$ are odd primes with $p_1 < \cdots < p_k$, $\ r_j \ (j = 0,\ 1,\ \cdots,\ k)$ are positive integers. By (10), we have

$$\varphi(x) = 2^{r_0 - 1} p_1^{r_1 - 1} \cdots p_k^{r_k - 1}(p_1 - 1)\cdots(p_k - 1). \tag{11}$$

Further, since x is a positive integer solution of equation (2), we see from (1) and (11) that

$$\varphi(x) = 2^{s_0} p_1^{s_1} \cdots p_k^{s_k}, \quad s_0 \geq r_0,\ s_i \geq \max(1,\ r_i - 1), \quad i = 1, \cdots, k. \tag{12}$$

Let

$$P = \left\{ m \mid m \in N^+, \quad m < \varphi(x), \quad m = 2^{t_0} p_1^{t_1} \cdots p_k^{t_k}, \quad t_j \in N^+, \quad j = 0, 1, \cdots, k \right\}. \tag{13}$$

Since $f(x^n) = \varphi(x)$, we see from (1) and (13) that

$$x^n \mid \varphi(x)^{\varphi(x)} \tag{14}$$

and

$$x^n \nmid m^m, \quad m \in P. \tag{15}$$

If $r_0 = 1, k = 1$ and $p_1 = 3$, then $x = 2 \times 3^{r_1}$ and $\varphi(x) = 2 \times 3^{r_1 - 1}$. By (14), we get

$$n = 2 \times 3^{r_1 - 1}\left(1 - \frac{1}{r_1}\right). \tag{16}$$

Further, since $n \geq 4$, we see from (16) that $r_1 \geq 3$. On the other hand, since $2 \times 3^{r_1 - 2} \in P$ by (13), we get from (15) that either

$$2 \times 3^{3^{r_1 - 2}} < n \tag{17}$$

or

$$2 \times 3^{3^{r_1 - 2}}\left(1 - \frac{2}{r_1}\right) < n. \tag{18}$$

Recall that $x = 2 \times 3^{r_1}$ and $r_1 \geq 3$. We deduce from (17) and (18) that $x < 27n$.

If $x \neq 2 \times 3^{r_1}$, then either $r_0 \geq 2$, $r_0 = k = 1$ and $p_1 \geq 5$ or $r_0 = 1$ and $k \geq 2$. It implies that $s_0 \geq 2$ by (10) and (12). Therefore, we see from (12) and (13) that $\frac{\varphi(x)}{2} \in P$. Further, by (10), (12) and (15), there exists a positive integer l such that $0 \leq l \leq k$ and

$$r_1 n > \begin{cases} \frac{1}{2}\varphi(x)(s_0 - 1), & \text{if } \ l = 0\ ; \\ \frac{1}{2}\varphi(x)(s_l), & \text{if } \ l \neq 0. \end{cases} \tag{19}$$

Furthermore, since $s_0 \geq \max(2,\ r_0)$ and $s_l \geq \max(1,\ r_l - 1)$ for $l \neq 0$, we get from (19) that

$$n > \frac{1}{4}\varphi(x). \tag{20}$$

Since $x > 2$, by Lemma 4, we have $\varphi(x) > \frac{x}{4\log x} > \frac{\sqrt{x}}{4}$. Hence, we see from (20) that $x < 256n^2$. To sum up, we obtain $x < \max(12n,\ 27n,\ 256n^2) = 256n^2$.

This completes the proof of Theorem 2.

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On the F.Smarandache $3n$-digital sequence

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Abstract The sequence $\{a_n\} = \{13, 26, 39, 412, 515, 618, 721, \cdots\}$ is called the F.Smarandache $3n$-digital sequence. That is, for any integer $b \in \{a_n\}$, it can be partitioned into two groups such that the second is three times bigger than the first. The main purpose of this paper is to study the mean value properties of $\frac{n}{a_n}$, and give an interesting mean value formula for it.

Keywords F.Smarandache $3n$-digital sequence, mean value, asymptotic formula.

§1. Introduction and result

For any positive integer n, the famous F.Smarandache $3n$-digital sequence is defined as $\{a_n\} = \{13, 26, 39, 412, 515, 618, 721, \cdots\}$. That is, for any integer $b \in \{a_n\}$, it can be divided into two parts such that the second is three times bigger than the first. For example, $a_{28} = 2884$, $a_{35} = 35105$, $a_{104} = 104312$, $\cdots$. This sequence was first proposed by professor F.Smarandache in references [1] and [4], where he also asked us to study its elementary properties. About this problem, some scholars had studied it, and obtained some interesting conclusions. For example, professor Zhang Wenpeng proposed the following:

Conjecture. There does not exist any complete square number in the Smarandache $3n$-digital sequence $\{a_n\}$. That is, the equation $a_n = m^2$ has no positive integer solution.

Wu nan [3] had studied this problem, and proved that $\{a_n\}$ is not a complete square number for some special positive integers n, such as complete square numbers and square-free numbers. For general positive integer n, whether Zhang's conjecture is true is an open problem.

In this paper, we shall use the elementary method to study the estimate properties of the mean value $\sum \frac{n}{a_n}$, and give a sharper asymptotic formula for it. That is, we shall prove the following conclusion:

Theorem. For any real number $N > 1$, we have the asymptotic formula

$$\sum_{n\leq N} \frac{n}{a_n} = \frac{3}{10\ln 10} \cdot \ln N + O\left(1\right).$$

§2. Proof of the theorem

In this section, we shall use the elementary method to complete the proof of our theorem. First for any positive integer number n, let $3n = b_{k(n)}b_{k(n)-1}\cdots b_2b_1$, where $1 \leq b_{k(n)} \leq 9$,

$0 \le b_i \le 9$, $i = 1,\ 2,\ \cdots,\ k(n)-1$. According to the definition of a_n, we can write a_n as

$$a_n = n \cdot 10^{k(n)} + 3n = n \cdot (10^{k(n)} + 3).$$

So we have

$$\sum_{n \le N} \frac{n}{a_n} = \sum_{n \le N} \frac{1}{10^{k(n)} + 3}.$$

It is clear that if $N \le 3$, then

$$\sum_{n \le N} \frac{n}{a_n} - \frac{3}{10} \log_{10} N$$

is a constant. So without loss of generality we can assume that $N > 3$. In this case, there exists a positive integer M such that

$$\underbrace{33 \cdots 33}_{M} < N \le \underbrace{33 \cdots 33}_{M+1}. \tag{1}$$

Note that for any positive integer n, if

$$\underbrace{33 \cdots 33}_{u-1} < n \le \underbrace{33 \cdots 33}_{u},$$

then $3n = b_u b_{u-1} \cdots b_2 b_1$. So we have

$$\begin{aligned}
\sum_{n \le N} \frac{n}{a_n} &= \sum_{n \le 3} \frac{1}{10+3} + \sum_{3 < n \le 33} \frac{1}{10^2+3} + \sum_{33 < n \le 333} \frac{1}{10^3+3} + \cdots \\
&\quad + \sum_{\underbrace{33\cdots33}_{M-1} < n \le \underbrace{33\cdots33}_{M}} \frac{1}{10^M+3} + \sum_{\underbrace{33\cdots33}_{M} < n \le N} \frac{1}{10^{M+1}+3} \\
&= \frac{3}{10+3} + \frac{30}{10^2+3} + \frac{300}{10^3+3} + \cdots + \frac{3 \cdot 10^{M-1}}{10^M+3} + \frac{N - \frac{10^M-1}{3}}{10^{M+1}+3} \\
&= \frac{3}{10}\left(\frac{10}{10+3} + \frac{10^2}{10^2+3} + \frac{10^3}{10^3+3} + \cdots + \frac{10^M}{10^M+3}\right) + \frac{N - \frac{10^M-1}{3}}{10^{M+1}+3} \\
&= \frac{3}{10}\left[\left(1 - \frac{3}{10+3}\right) + \left(1 - \frac{3}{10^2+3}\right) + \left(1 - \frac{3}{10^3+3}\right) + \cdots + \left(1 - \frac{3}{10^M+3}\right)\right] \\
&\quad + \frac{N - \frac{10^M-1}{3}}{10^{M+1}+3} \\
&= \frac{3}{10}\left[M - \left(\frac{3}{10+3} + \frac{3}{10^2+3} + \frac{3}{10^3+3} + \cdots + \frac{3}{10^M+3}\right)\right] + \frac{N - \frac{10^M-1}{3}}{10^{M+1}+3} \\
&= \frac{3}{10} \cdot M - \frac{9}{10} \cdot \sum_{i=1}^{M} \frac{1}{10^i+3} + \frac{N - \frac{10^M-1}{3}}{10^{M+1}+3} \\
&= \frac{3}{10} \cdot M + O(1).
\end{aligned} \tag{2}$$

Now we estimate M, from inequality (1) we have

$$10^M - 1 < 3N \le 10^{M+1} - 1$$

$$M\ln 10+\ln\left(1-\frac{1}{10^M}\right)<\ln(3N)\le (M+1)\ln 10+\ln\left(1-\frac{1}{10^{M+1}}\right)$$

$$\frac{\ln(3N)}{\ln 10}-\frac{\ln(1-\frac{1}{10^{M+1}})}{\ln 10}-1\le M<\frac{\ln(3N)}{\ln 10}-\frac{\ln(1-\frac{1}{10^{M}})}{\ln 10}.$$

Note that as $N\to+\infty$, $\ln(1-\frac{1}{10^{M+1}})\sim\frac{1}{10^M}$, $\ln(1-\frac{1}{10^{M}})\sim\frac{1}{10^M}$. So that

$$\frac{\ln 3N}{\ln 10}-1-O\left(\frac{1}{10^M}\right)\le M<\frac{\ln 3N}{\ln 10}-O\left(\frac{1}{10^M}\right).$$

Combining this and (2) we may immediately deduce that

$$\sum_{n\le N}\frac{n}{a_n}=\frac{3}{10\ln 10}\cdot\ln N+O\left(1\right).$$

This completes the proof of Theorem.

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An equation involving the Smarandache double factorial function and Euler function

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Abstract For any positive integer n, let $Sdf(n)$ denotes the Smarandache double factorial function, and $\varphi(n)$ is the Euler function. The main purpose of this paper is using the elementary method to study the solvability of the equation $Sdf(n) = \varphi(n)$, and give its all positive integer solutions.

Keywords Double factorial function, Euler function, equation, positive integer solutions.

§1. Introduction and results

For any positive integer n, the famous Smarandache double factorial function $Sdf(n)$ is defined as the smallest positive integer m such that $m!!$ is divisible by n, where the double factorial

$$m!! = \begin{cases} 1 \cdot 3 \cdot 5 \cdots m, & \text{if } m \text{ is an odd number ;} \\ 2 \cdot 4 \cdot 6 \cdots m, & \text{if } m \text{ is an even number.} \end{cases}$$

That is, $Sdf(n) = \min\{m : n|m!!,\ m \in N\}$, where N denotes the set of all positive integers. For example, the first few values of $Sdf(n)$ are: $Sdf(1) = 1$, $Sdf(2) = 2$, $Sdf(3) = 3$, $Sdf(4) = 4$, $Sdf(5) = 5$, $Sdf(6) = 6$, $Sdf(7) = 7$, $Sdf(8) = 4$, $Sdf(9) = 9$, $Sdf(10) = 10$, $Sdf(11) = 11$, $Sdf(12) = 6$, $Sdf(13) = 13$, $Sdf(14) = 14$, $Sdf(15) = 5$, $Sdf(16) = 6$, $Sdf(17) = 17$, $Sdf(18) = 12$, $Sdf(19) = 19$, $Sdf(20) = 10$, $\cdots$. In reference [1] and [2], Professor F.Smarandache asked us to study the properties of $Sdf(n)$. About this problem, some authors had studied it, and obtained some interesting results, see references [3], [4], [5] and [6]. For example, Maohua Le [4] discussed various problems and conjectures about $Sdf(n)$, and obtained some useful results, one of them as follows: if $2|n$ and $n = 2^\alpha n_1$, where $\alpha,\ n_1$ are positive integers with $2 \nmid n_1$, then

$$Sdf(n) \le \max\{Sdf(2^\alpha),\ 2Sdf(n_1)\}.$$

Fuling Zhang and Jianghua Li [5] proved that for any real number $x \ge 1$, we have the asymptotic formula

$$\sum_{n \le x} Sdf(n) = \frac{x \ln x}{\ln \ln x} + O\left(\frac{x \ln x}{(\ln \ln x)^2}\right).$$

Jianping Wang [6] proved that for any real number $x \geq 1$ and any fixed positive integer k, we have the asymptotic formula

$$\sum_{n\leq x}(Sdf(n)-S(n))^2 = \frac{\zeta(3)}{24}\frac{x^3}{\ln x} + \sum_{i=2}^{k}\frac{c_i \cdot x^3}{\ln^i x} + O\left(\frac{x^3}{\ln^{k+1} x}\right),$$

where $\zeta(s)$ is the Riemann zeta-function, and c_i are constants.

In this paper, we shall use the elementary method to study the solvability of the equation $Sdf(n) = \varphi(n)$, and give its all positive integer solutions. That is, we shall prove the following conclusion:

Theorem. The equation $Sdf(n) = \varphi(n)$ have only 4 positive integer solutions, they are $n = 1, 8, 36, 50$.

§2. Proof of the theorem

In this section, we shall use the elementary method to complete the proof of the theorem. It is easy to versify that $n = 1$ is a solution of the equation $Sdf(n) = \varphi(n)$. In order to obtain the other positive integer solutions, we discuss the equation in the following several cases:

1. If $n > 1$ is an odd number. At this time, from the definition of the Smarandache double factorial function $Sdf(n)$ we know that $Sdf(n)$ is an odd number, but $\varphi(n)$ is an even number, thus $Sdf(n) \neq \varphi(n)$.

2. If n is an even number. We assume that $n = 2^\alpha p_1^{\alpha_1} p_2^{\alpha_2}\cdots p_k^{\alpha_k}$, where $p_1 < p_2 < \cdots < p_k$, p_i $(1 \leq i \leq k)$ is an odd prime, $\alpha_i \geq 0$ $(1 \leq i \leq k)$, $\alpha \geq 1$.

(1) If $\alpha_i = 0$ $(1 \leq i \leq k)$, then $n = 2^\alpha$ $(\alpha \geq 1)$. It is easy to versify that $n = 2,\ 2^2,\ 2^4$ are not solutions of the equation $Sdf(n) \neq \varphi(n)$, and $n = 2^3$ is a solution of the equation $Sdf(n) = \varphi(n)$. If $\alpha \geq 5$, since $n \mid \varphi(n)!!$, so we obtain $2^\alpha \mid 2{\cdot}2^2\cdots 2^{\alpha-2}$, namely $2^\alpha \mid \left(\frac{\varphi(n)}{2}\right)!!$, which implies $Sdf(n) < \frac{\varphi(n)}{2} < \varphi(n)$.

(2) If $\alpha_i \geq 1$,

(I) $n = 2^\alpha p_1^{\alpha_1} p_2^{\alpha_2}\cdots p_k^{\alpha_k}$, where $p_1 < p_2 < \cdots < p_k$, p_i $(1 \leq i \leq k)$ is an odd prime, $\alpha_i \geq 1$ $(1 \leq i \leq k)$, $\alpha \geq 2,\ k \geq 1$. At this time,

$$\varphi(n) = 2^{\alpha-1}p_1^{\alpha_1-1}p_2^{\alpha_2-1}\cdots p_k^{\alpha_k-1}(p_1-1)(p_2-1)\cdots(p_k-1).$$

If $n \nmid \varphi(n)!!$, then from the definition of the Smarandache double factorial function $Sdf(n)$ we know that $Sdf(n) \neq \phi(n)$.

If $n \mid \varphi(n)!!$, then

(i) For $2^\alpha \mid n$, $(\alpha \geq 2)$, $\frac{\varphi(n)}{2} = 2^{\alpha-1}p_1^{\alpha_1-1}p_2^{\alpha_2-1}\cdots p_k^{\alpha_k-1}(p_1-1)(p_2-1)\cdots\frac{(p_k-1)}{2}$.

If $\alpha_k = 1,\ p_k = 3$, then $n = 2^\alpha \cdot 3$, $\varphi(n) = 2^\alpha$. Since $n \mid \varphi(n)!!$, so we have $\alpha > 2$. Thus $2^\alpha \mid \left(\frac{\varphi(n)}{2}\right)!!$.

If $\alpha_k > 1,\ p_k \geq 3$ or $\alpha_k = 1,\ p_k > 3$, then $4 \leq p_k^{\alpha_k-1}(p_k-1)$, so we have $2^\alpha \mid \left(\frac{\varphi(n)}{2}\right)!!$.

(ii) For $p_i^{\alpha_i} \mid n$, $(1 \leq i \leq k)$.

(a) If $\alpha_i = 1,\ (1 \le i \le k)$, since $n \mid \varphi(n)!!$, we can deduce that

$$2p_k \le 2^{\alpha-1}(p_1-1)(p_2-1)\cdots(p_k-1).$$

$(a)'$ If $k=1$, then $\alpha > 2$. If $\alpha = 3$, then $n = 2^3 \cdot p_k$. According to the definition of the Smarandache double factorial function $Sdf(n)$ we have

$$Sdf(2^3 \cdot p_k) \le \max\{Sdf(2^3),\ 2Sdf(p_k)\} \le 2\cdot p_k < 4(p_k-1) = \varphi(2^3\cdot p_k).$$

At this time, the equation $Sdf(n) = \varphi(n)$ have no positive integer solution.

If $\alpha > 3$, then from $4p_k \le 2^{\alpha-1}(p_k-1)$ we have $p_k \mid \left(\dfrac{\varphi(n)}{2}\right)!!$.

$(b)'$ If $k=2$, then $\alpha \ge 2$. If $\alpha = 2,\ p_1 = 3$, then from the definition of the Smarandache double factorial function $Sdf(n)$ we can deduce that

$$Sdf(2^2\cdot 3\cdot p_k) \le \max\{Sdf(2^2),\ 2Sdf(p_k)\} \le 2\cdot p_k < 4(p_k-1) = \varphi(2^2\cdot 3\cdot p_k).$$

At this time, the equation $Sdf(n) = \varphi(n)$ have no positive integer solution.

If $\alpha \ge 3$ or $\alpha = 2,\ p_1 > 3$, then $4p_k \le 2^{\alpha-1}(p_1-1)(p_k-1)$, so we have $p_k \mid \left(\dfrac{\varphi(n)}{2}\right)!!$.

$(c)'$ If $k > 2$, then $\alpha \ge 2$. From $4p_k \le 2^{\alpha-1}(p_1-1)(p_2-1)\cdots(p_k-1)$, we can obtain $p_k \mid \left(\dfrac{\varphi(n)}{2}\right)!!$.

(b) If $\alpha_i \ge 2,\ (1\le i\le k)$, then from $n \mid \varphi(n)!!$ we have $p_i^{\alpha_i} \mid \varphi(n)!!$, thus $2p_i < \varphi(n)$.

$(a)'$ If $k=1$, then $n = 2^\alpha\cdot p_k^{\alpha_k}$, $\varphi(2^\alpha\cdot p_k^{\alpha_k}) = 2^{\alpha-1}\cdot p_k^{\alpha_k-1}\cdot(p_k-1)$.

If $\alpha_k = 2$ and $\alpha = 2$, then $n = 2^2\cdot p_k^2$. If $p_k = 3$, according to the definition of the Smarandache double factorial function $Sdf(n)$ we can easily obtain $Sdf(2^2\cdot 3^2) = 12 = \varphi(2^2\cdot 3^2)$, so $n = 36$ is another positive integer solution of the equation $Sdf(n) = \varphi(n)$. If $p_k > 3$, then from the definition of the Smarandache double factorial function $Sdf(n)$ we obtain

$$Sdf(2^2\cdot p_k^2) \le \max\{Sdf(2^2),\ 2Sdf(p_k^2)\} = 2\cdot 3\cdot p_k < 2\cdot p_k\cdot(p_k-1) = \varphi(2^2\cdot p_k^2).$$

At this time, the equation $Sdf(n) = \varphi(n)$ have no positive integer solution.

If $\alpha = 2,\ \alpha_k > 2$ or $\alpha > 2,\ \alpha_k = 2$ or $\alpha > 2,\ \alpha_k > 2$, then from $4p_k < 2^{\alpha-1}p_k^{\alpha_k-1}(p_k-1)$ we have $4p_k < \varphi(2^\alpha\cdot p_k^{\alpha_k})$, thus $p_k^{\alpha_k} \mid \left(\dfrac{\varphi(n)}{2}\right)!!$.

$(b)'$ If $k \ge 2$, then from $n \mid \varphi(n)!!$ we have $4p_i < 2^{\alpha-1}p_1^{\alpha_1-1}\cdots p_i^{\alpha_i-1}\cdots p_k^{\alpha_k-1}(p_1-1)\cdots(p_i-1)\cdots(p_k-1)$, thus $p_i^{\alpha_i} \mid \left(\dfrac{\varphi(n)}{2}\right)!!$.

Combining (i) and (ii) we can deduce that:

If $n = 36$, then $Sdf(n) = \varphi(n)$.

If $n = 2^3\cdot p_k$, $n = 2^2\cdot 3\cdot p_k$ or $n = 2^2\cdot p_k^2$ $(p_k > 3)$, then $Sdf(n) < \varphi(n)$.

Otherwise, from $n \mid \varphi(n)!!$, we have $Sdf(n) \le \dfrac{\varphi(n)}{2} < \varphi(n)$.

(II) $n = 2p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k}$, where $p_1 < p_2 < \cdots < p_k$, p_i $(1\le i\le k)$ is an odd prime, $\alpha_i \ge 1$ $(1\le i\le k)$, $k\ge 1$. At this time,

$$\varphi(n) = p_1^{\alpha_1-1}p_2^{\alpha_2-1}\cdots p_k^{\alpha_k-1}(p_1-1)(p_2-1)\cdots(p_k-1).$$

If $n \nmid (\varphi(n))!!$, then from the definition of the Smarandache double factorial function $Sdf(n)$ we know that $Sdf(n) \neq \varphi(n)$.

If $n \mid \varphi(n)!!$, then

(i) If $k = 1$, then $n = 2p_k^{\alpha_k}$, $\varphi(n) = p_k^{\alpha_k - 1}(p_k - 1)$. From $2p_k^{\alpha_k} \mid (p_k^{\alpha_k - 1}(p_k - 1))!!$, we have $2p_k < p_k^{\alpha_k - 1}(p_k - 1)$. Hence $p_k = 3,\ \alpha_k \geq 3$ or $p_k \geq 5,\ \alpha_k \geq 2$.

(a) If $p_k = 3,\ \alpha_k \geq 3$, then from $2 \cdot 3 < 2 \cdot 3^{\alpha_k - 1}$, so we have $4 \cdot 3 < 2 \cdot 3^{\alpha_k - 1}$. Thus $2p_k^{\alpha_k} \mid \left(\dfrac{\varphi(2p_k^{\alpha_k})}{2}\right)!!$.

(b) If $p_k = 5,\ \alpha_k = 2$, then $n = 2 \cdot 5^2$. According to the definition of the Smarandache double factorial function $Sdf(n)$ we can easily obtain $Sdf(2 \cdot 5^2) = 20 = \varphi(2 \cdot 5^2)$, so $n = 50$ is another solution of the equation $Sdf(n) = \varphi(n)$.

(c) If $p_k = 5,\ \alpha_k > 2$ or $p_k > 5,\ \alpha_k \geq 2$, then we can deduce that $4p_k < p_k^{\alpha_k - 1}(p_k - 1)$. Thus $2p_k^{\alpha_k} \mid \left(\dfrac{\varphi(2p_k^{\alpha_k})}{2}\right)!!$.

(ii) If $k \geq 2$, then from $p_i^{\alpha_i} \mid \varphi(n)!!$, so we have $2p_i < p_1^{\alpha_1 - 1} \cdots p_i^{\alpha_i - 1} \cdots p_k^{\alpha_k - 1}(p_1 - 1) \cdots (p_i - 1) \cdots (p_k - 1)$, hence there at least exists a α_i such that $\alpha_i \geq 2$, so we can deduce that $4p_i < p_1^{\alpha_1 - 1} \cdots p_i^{\alpha_i - 1} \cdots p_k^{\alpha_k - 1}(p_1 - 1) \cdots (p_i - 1) \cdots (p_k - 1)$ which implies $p_i^{\alpha_i} \mid \left(\dfrac{\varphi(n)}{2}\right)!!$.

Combining (i) and (ii), we can deduce that:

If $n = 50$, then $Sdf(n) = \varphi(n)$.

Otherwise, from $n \mid \varphi(n)!!$, we have $n \mid \left(\dfrac{\varphi(n)}{2}\right)!!$ which implies $Sdf(n) \leq \dfrac{\varphi(n)}{2} < \varphi(n)$.

From the above discussion (1) and (2), we know that if n is an even number, then $Sdf(n) = \varphi(n)$ if and only if $n = 8,\ 36,\ 50$.

Combining the cases 1 and 2, we complete the proof of Theorem.

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S-perfect and completely S-perfect numbers

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Abstract This paper proves that the only Smarandache perfect numbers are $n = 1,\ 6$, and the only completely Smarandache perfect numbers are $n = 1,\ 28$, where the Smarandache function $S(n)$. satisfies the condition that $S(1) = 1$.

Keywords The Smarandache function, the Smarandache perfect number, the completely Smarandache perfect number.

§1. Introduction

The Smarandache function is defined as follows.

Definition 1.1. For any integer $n \geq 1$, the Smarandache function $S(n)$, is the smallest positive integer m such that $1 \cdot 2 \cdot \cdots \cdot m \equiv m!$ is divisible by n. That is,

$$S(n) = \min\ \{m : m \in \mathbb{Z}^+,\ n \mid m!\};\ n \geq 1,$$

where $S(1) = 1$ (and $\mathbb{Z}^+$ is the set of all positive integers).

In classical Theory of Numbers, an integer $n \geq 1$ is called perfect if it is the sum of its proper divisors. Pe [1] has extended the definition to f - perfect numbers : If $f(n)$ is an arithmetical function, an integer n with proper divisors $d_1 \equiv 1, d_2, \cdots, d_k$ is called f-perfect if

$$n = \sum_{i=1}^{k} f(d_i).$$

Further extension is the completely f - perfect number n where the condition of proper divisor is relaxed.

The S-perfect and completely S-perfect numbers are defined below.

Definition 1.2. Given an integer $n \geq 1$,

(1) n is called Smarandache perfect (or, simply S-perfect) if and only if $n = \sum\limits_{i=1}^{k} S(d_i)$, (where $d_1 \equiv 1, d_2, \cdots, d_k$ are the proper divisors of n);

(2) n is called completely S-perfect if and only if $n = \sum\limits_{d|k} S(d)$, (where the sum is over all divisors d of n (including n)).

In this paper, we prove that the only S-perfect numbers are $n = 1, 6$, and the only completely S-perfect numbers are $n = 1, 28$. These are given in Theorem 2.1 and Theorem 2.2 in Section 2. We conclude this paper with some remarks in the final section, Section 3.

§2. S-perfect and completely S - perfect numbers

In this section, we derive the numbers that are S-perfect, and the numbers that are completely S-perfect. They are given in the following two theorems.

Theorem 2.1. The only S-perfect numbers are $n = 1, 6$.

Proof. Let n be an S-perfect number, so that, by Definition 1.2,

$$n = \sum_{\substack{d \mid n \\ 1 \le d < n}} S(d). \tag{1}$$

Clearly, no prime is a solution of (1). Thus, any solution of (1) must be composite a number. Now, for any divisor d of n, $S(d) \le S(n)$, with strict inequality sign for at least one divisor of n. Therefore, from (1), we get

$$n = \sum_{\substack{d \mid n \\ 1 \le d < n}} S(d) < [d(n) - 1]S(n) \ \Rightarrow \ n < d(n)S(n). \tag{2}$$

Thus, any solution n of (1) must satisfy the inequalities in (2).

We first consider some particular cases.

Case 1. When n is of the form $n = p^\alpha$ $(\alpha \ge 2)$.

In this case, $d(n) = \alpha + 1$, so that (2) reads as

$$n = p^\alpha < \alpha S(p^\alpha) \le \alpha.(\alpha p) \ \Rightarrow \ p^{\alpha-1} < \alpha^2. \tag{3}$$

But (3) is impossible when (a) $p = 2$ and $\alpha \ge 7$, (b) $p \ge 3$ for any $\alpha \ge 3$.

The proof is as follows :

(a) $\dfrac{2^{\alpha-1}}{\alpha^2}$ is strictly increasing in $\alpha \ge 7$ with $\dfrac{2^{\alpha-1}}{\alpha^2} \ge \dfrac{2^6}{7^2} > 1$,

(b) for any $p \ge 3$, $\dfrac{p^{\alpha-1}}{\alpha^2}$ is strictly increasing in $\alpha \ge 3$ with $\dfrac{p^{\alpha-1}}{\alpha^2} \ge \dfrac{3^2}{3^2} = 1$.

Thus, in this case, the possible candidates for the equation (1) to hold true are

$$n = 2^\alpha, 2 \le \alpha \le 6; n = 3^2.$$

But, when $n = 2^\alpha$, the proper divisors of n are $1, 2, 2^2, \cdots 2^{\alpha-1}$, with $S(1) = 1$, so that the r.h.s. of (1) is odd, while n is even. Again, when $n = 3^2$, r.h.s. of (1) is not divisible by 3. Thus, in this case, there is no solution of the equation (1).

Case 2. When n is of the form $n = p\,q$ with $q > p$.

In this case, $d(n) = 4$, and so, from (2) we get

$$n = pq < 3q \ \Rightarrow \ p = 2.$$

Now, if $p = 2$, then

$$\sum_{\substack{d|n \\ 1 \le d < n}} S(d) = S(1) + S(p) + S(q) = 1 + p + q = 3 + q = n = 2q \Rightarrow q = 3.$$

Thus, in this case, $n = 2 \cdot 3 = 6$ is the only solution of the equation (1).

Case 3. When n is of the form $n = 2^\alpha q, q \geq 3, \alpha \geq 2$.

First, let $S(n) = S(2^\alpha)$(so that $S(2^\alpha \geq q)$. Since $d(n) = 2(\alpha+1)$, from(2), we get

$$n = 2^\alpha q < 2(\alpha+1)S(2^\alpha) \leq 2^2\alpha(\alpha+1) \;\Rightarrow\; 2^{\alpha-2}q < \alpha(\alpha+1),$$

which is impossible if (i) $q = 3$ and $\alpha \geq 6$, or if (ii) $q = 5$ and$\alpha \geq 4$, or if (iii) $q \geq 7$ and $\alpha \geq 2$. Thus, the possible candidates for the equation (1) to hold true are

$$n = 3.2^\alpha, 2 \leq \alpha \leq 5; (n = 5.2^3 \text{ is excluded since } S(2^3) = 4 < q = 5).$$

Now, when $n = 2^2q$, the proper divisors of n are $1, 2, 2^2, q$ and $2q$, so that

$$\sum_{\substack{d|n \\ 1\leq d<n}} S(d) = S(1) + S(2) + S(2^2) + S(q) + S(2q) = 1 + 2 + 4 + q + q = 7 + 2q,$$

when $n = 2^3q$, the proper divisors of n are $1, 2, 2^2, 2^3, q, 2q$ and 2^2q, and so

$$\begin{aligned} \sum_{\substack{d|n \\ 1\leq d<n}} S(d) &= S(1) + S(2) + S(2^2) + S(2^3) + S(q) + S(2q) + S(2^2q) \\ &= 1 + 2 + 4 + 4 + q + q + 4 + 4 = 15 + 2q, \end{aligned}$$

when $n = 2^4q$, the proper divisors of n are $1, 2, 2^2, 2^3, 2^4, q, 2q, 2^2q$ and 2^3q, so that

$$\begin{aligned} \sum_{\substack{d|n \\ 1\leq d<n}} S(d) &= S(1) + S(2) + S(2^2) + S(2^3) + S(2^4) + S(q) + S(2q) + S(2^2q) + S(2^3q) \\ &= 1 + 2 + 4 + 4 + 6 + q + q + 4 = 25 + 2q, \end{aligned}$$

the proper divisors of $n = 2^5q$ are $1, 2, 2^2, 2^3, 2^4, 2^5, q, 2q, 2^2q, 2^3q$ and 2^4q, with

$$\begin{aligned} \sum_{\substack{d|n \\ 1\leq d<n}} S(d) &= S(1) + S(2) + S(2^2) + S(2^3) + S(2^4) \\ &+ S(2^5) + S(q) + S(2q) + S(2^2q) + S(2^3q) + S(2^4q) \\ &= 1 + 2 + 4 + 4 + 6 + 8 + q + q + 4 + 4 + 6 = 39 + 2q, \end{aligned}$$

so that, in each case, the term on the right of (1) is odd, while n is even.
Next, let $S(n) = q$. Then, from (2), we get $n = 2^\alpha q < 2(\alpha+1)q \;\Rightarrow\; 2^{\alpha-1} < \alpha+1$, which is possible only for $\alpha = 2$, and so,$n = 2^2q$. And we have just proved that $n = 2^2q$ is not a solution of (1). Thus, in this case, there is no solution of the equation (1).

Case 4. When n is of the form $n = p^\alpha q^\beta(\alpha \geq 1, \beta \geq 1, \alpha\beta \geq 2)$.

Let $S(n) = S(p^\alpha)$. Now, since $d(n) = (\alpha+1)(\beta+1)$, (2)takes the form

$$n = p^\alpha q^\beta < (\alpha+1)(\beta+1)S(p^\alpha) \leq (\alpha+1)(\beta+1)\alpha p \Rightarrow \frac{p^{\alpha-1}}{\alpha(\alpha+1)} \cdot \frac{q^\beta}{\beta+1} < 1. \qquad (4)$$

Note that, since $\dfrac{p^{\alpha-1}}{\alpha(\alpha+1)} > 1$ for any $p \geq 7$ and any $\alpha \geq 2$ and $\dfrac{q^{\beta}}{\beta+1} \geq 1$ for any $q \geq 2$ and any $\beta \geq 1$, (4) is impossible for any $q \geq 7$, any $q \neq p$, any $\alpha \geq 2$ and any $\beta \geq 1$. Thus, it is sufficient to check with $p = 2, 3, 5$ only.

First, let $p = 2$(so that $q \geq 3, \alpha \geq \beta \geq 2$). Then, (4) reads as

$$\frac{2^{\alpha-1}}{\alpha(\alpha+1)} \cdot \frac{q^{\beta}}{\beta+1} < 1. \tag{5}$$

If $q \geq 3$ and $\beta \geq 2$, then since $\dfrac{q^{\beta}}{\beta+1} \geq 3$, we get from (5), $\dfrac{3 \cdot 2^{\alpha-1}}{\alpha(\alpha+1)} < 1$,which is impossible for any $\alpha \geq 1$. Thus, (1) has no solution corresponding to this case (by Case 3 above).
Next, let p = 3, so that from (4)

$$\frac{3^{\alpha-1}}{\alpha(\alpha+1)} \cdot \frac{q^{\beta}}{\beta+1} < 1. \tag{6}$$

If $q \geq 5$ and $\beta \geq 1$,then since $\dfrac{q^{\beta}}{\beta+1} > 2$, we get from (6), $\dfrac{2 \cdot 3^{\alpha-1}}{\alpha(\alpha+1)} < 1$, which is impossible for any $\alpha \geq 1$; if $q = 2$ and $\beta \geq 3$, then since $\dfrac{2^{\beta}}{\beta+1} \geq 2$, from (6), $\dfrac{2 \cdot 3^{\alpha-1}}{\alpha(\alpha+1)} < 1$, which is impossible for any $\alpha \geq 1$; moreover, from (6), $\dfrac{3^{\alpha-1}}{\alpha(\alpha+1)} < 1$, which is impossible for $\alpha \geq 4$.
Thus, in this case, the only possible candidates for (1) to hold true are

$$n = 2 \cdot 3^{\alpha}, 2^2 \cdot 3^{\alpha} \text{ where } 2 \leq \alpha \leq 3.$$

But, $n = 2 \cdot 3^{\alpha}$ has the proper divisors $1, 2, 3, 3^2, \cdots, 3^{\alpha}, 2 \cdot 3, 2 \cdot 3^2, \cdots, 2 \cdot 3^{\alpha-1}$, so that

$$\begin{aligned}\sum_{\substack{d|n\\1\leq d<n}} S(d) &= S(1)+S(2)+S(3)+S(3^2)+\cdots+S(3^{\alpha})+S(2\cdot 3)+S(2\cdot 3^2)+\cdots+S(2\cdot 3^{\alpha-1})\\ &= 1+2+\sum_{k=1}^{\alpha}(3k)+\sum_{k=1}^{\alpha-1}(3k) = 1+2+3\frac{\alpha(\alpha+1)}{2}+3\frac{\alpha(\alpha-1)}{2} = 3(\alpha^2+1),\end{aligned}$$

and the equation $3(\alpha^2 + 1) = 2 \cdot 3^{\alpha}$ has no solution for $\alpha \geq 2$.
Again, since $1, 2, 2^2, 3, 3^2, \cdots, 3^{\alpha}, 2.3, 2 \cdot 3^2, \cdots, 2 \cdot 3^{\alpha}, 2^2 \cdot 3, 2^2 \cdot 3^2, \cdots, 2^2 \cdot 3^{\alpha-1}$ are the proper divisors of $n = 2^2 \cdot 3^{\alpha}$, we get

$$\begin{aligned}\sum_{\substack{d|n\\1\leq d<n}} S(d) &= S(1)+S(2)+S(2^2)+S(3)+S(3^2)+\cdots+S(3^{\alpha})\\ &+ S(2\cdot 3)+S(2\cdot 3^2)+\cdots+S(2\cdot 3^{\alpha})+S(2^2\cdot 3)+S(2^2\cdot 3^2)+\cdots+S(2^2\cdot 3^{\alpha-1})\\ &= 1+2+4+2\sum_{k=1}^{\alpha}(3k)+4+\sum_{k=2}^{\alpha-1}(3k),\end{aligned}$$

which is not divisible by 3.

Finally, let $p = 5$, so that (4) takes the form

$$\frac{5^{\alpha-1}}{\alpha(\alpha+1)} \cdot \frac{q^\beta}{\beta+1} < 1. \tag{7}$$

If $q \geq 7$ and $\beta \geq 1$, then since $\dfrac{q^\beta}{\beta+1} > 3$, we get from (7), $\dfrac{3.5^{\alpha-1}}{\alpha(\alpha+1)} < 1$, which is impossible for any $\alpha \geq 1$; if $q = 2$ and $\beta \geq 3$, then since $\dfrac{2^\beta}{\beta+1} \geq 2$, from (7), $\dfrac{2 \cdot 5^{\alpha-1}}{\alpha(\alpha+1)} < 1$, which is impossible for any $\alpha \geq 1$; if $q = 3$ and $\beta \geq 2$, then since $\dfrac{3^\beta}{\beta+1} \geq 3$, from (7), $\dfrac{3 \cdot 5^{\alpha-1}}{\alpha(\alpha+1)} < 1$, which is impossible for any $\alpha \geq 1$; moreover, from (7), $\dfrac{5^{\alpha-1}}{\alpha(\alpha+1)} < 1$, which is impossible for $\alpha \geq 3$. Thus, in this case, the only possible candidates for (1) to hold true are $n = 3 \cdot 5^2, 2 \cdot 5^2, 2^2 \cdot 5^2$.

Note, however, that $n = 3 \cdot 5^2$ violates (7), since $\dfrac{5}{6} \cdot \dfrac{3}{2} > 1$.

Now, when $n = 2 \cdot 5^2$, the proper divisors of n are $1, 2, 5, 5^2$ and 2.5, so that

$$\sum_{\substack{d|n \\ 1 \leq d < n}} S(d) = S(1) + S(2) + S(5) + S(5^2) + S(2.5) = 1 + 2 + 5 + 10 + 5 = 23,$$

and the proper divisors of $n = 2^2 \cdot 5^2$ are $1, 2, 2^2, 5, 5^2, 2 \cdot 5, 2^2 \cdot 5$ and $2 \cdot 5^2$, with

$$\sum_{\substack{d|n \\ 1 \leq d < n}} S(d) = S(1) + S(2) + S(2^2) + S(5) + S(5^2) + S(2 \cdot 5) + S(2^2 \cdot 5) + S(2 \cdot 5^2) = 42.$$

Thus, there is no solution of (1) of the form $n = p^\alpha q^\beta$.
Now, we consider the general case. So, let

$$n = p^\alpha q^\beta p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$$

be the representation of n in terms of its distinct prime factors $p, q, p, p_1, p_2, \cdots, p_k (k \geq 1)$.

Let, for definiteness, $S(n) = S(p^\alpha)$. Then, from (2), we get

$$n < d(n)S(n) \leq d(n)S(p^\alpha) \Rightarrow \frac{p^\alpha}{\alpha+1} \cdot \frac{q^\beta}{\beta+1} \cdot \frac{p_1^{\alpha_1}}{\alpha_1+1} \frac{p_2^{\alpha_2}}{\alpha_2+1} ... \frac{p_k^{\alpha_k}}{\alpha_k+1} < S(p^\alpha) \leq \alpha p. \tag{8}$$

Now, if $p = 2$, without loss of generality, $q \geq 5$, and so (8) takes the form

$$\frac{2^\alpha}{\alpha(\alpha+1)} \cdot \frac{p_1^{\alpha_1}}{\alpha_1+1} \cdot \frac{p_2^{\alpha_2}}{\alpha_2+1} \cdots \frac{p_k^{\alpha_k}}{\alpha_k+1} < 1,$$

which is impossible. On the other hand, if $p = 3$, with $q \geq 5$, (8) takes the form

$$\frac{2.3^{\alpha-1}}{\alpha(\alpha+1)} \cdot \frac{p_1^{\alpha_1}}{\alpha_1+1} \cdot \frac{p_2^{\alpha_2}}{\alpha_2+1} \cdots \frac{p_k^{\alpha_k}}{\alpha_k+1} < 1,$$

which is also impossible.

All these complete the proof of the theorem.

Theorem 2.2. The only completely S-perfect numbers are $n = 1,\ 28$.

Proof. Let n be a completely S-perfect number, so that, by Definition 2.1,

$$n = \sum_{\substack{d|n \\ 1 \le d \le n}} S(d). \tag{9}$$

Clearly, no prime is a solution of (9). Thus, any solution of (9) must be composite a number.

In this case, any solution of (9) must satisfy the following condition:

$$n = \sum_{\substack{d|n \\ 1 \le d \le n}} S(d) < d(n)S(n). \tag{10}$$

As in the proof of Theorem 2.1, in this case also, we consider the following case:

Case 1. When n is of the form $n = p^{\alpha} (\alpha \ge 2)$.

In this case, however, there is no solution of the equation (9)(by an argument similar to that given in the proof of Theorem 2.1).

Case 2. When n is of the form $n = pq$ with $q \ge p$.

In this case,

$$\sum_{\substack{d \,|\, n \\ 1 \le d \le n}} S(d) = S(1) + S(p) + S(q) + S(pq) = 1 + p + q + q = 1 + p + 2q,$$

which is odd only if $p = 2$. But, then

$$\sum_{\substack{d|n \\ 1 \le d \le n}} S(d) = 3 + 2q > 2q.$$

Thus, there is no solution in this case.

Case 3. When n is of the form $n = 2^{\alpha} q, q \ge 3, \alpha \ge 2$.

Letting $S(n) = S(2^{\alpha})$, the possible candidates for (9) to hold true are $n = 3.2^{\alpha}, 2 \le \alpha \le 5$. But,

(**a**) When $n = 2^2 q$, $\displaystyle\sum_{\substack{d|n \\ 1 \le d \le n}} S(d) = \sum_{\substack{d|n \\ 1 \le d < n}} S(d) + S(2^2 q) = 11 + 2q,$

(**b**) When $n = 2^3 q$, $\displaystyle\sum_{\substack{d|n \\ 1 \le d \le n}} S(d) = \sum_{\substack{d|n \\ 1 \le d < n}} S(d) + S(2^3 q) = 19 + 2q,$

(**c**) When $n = 2^4 q$, $\displaystyle\sum_{\substack{d|n \\ 1 \le d \le n}} S(d) = \sum_{\substack{d|n \\ 1 \le d < n}} S(d) + S(2^4 q) = 31 + 2q,$

(**d**) When $n = 2^5 q$, $\displaystyle\sum_{\substack{d|n \\ 1 \le d \le n}} S(d) = \sum_{\substack{d|n \\ 1 \le d < n}} S(d) + S(2^5 q) = 47 + 2q,$

so that in all the cases, the r.h.s. of (9) is odd.
Next, assuming that $S(n) = q$, we get from (10),

$$n = 2^\alpha q < 2(\alpha+1)q \quad \Rightarrow \quad 2^{\alpha-1} < \alpha+1,$$

which is possible only for $\alpha = 2$, and so, $n = 2^2 q$. Then,

$$\begin{aligned}\sum_{\substack{d \,\mid\, n \\ 1 \le d \le n}} S(d) &= S(1) + S(2) + S(2^2) + S(q) + S(2q) + S(2^2 q) \\ &= 1 + 2 + 4 + q + q + q = 3q + 7 = 2^2 q \\ &\Rightarrow \quad q = 7.\end{aligned}$$

Thus, in this case, $n = 2^2.7 = 28$ is the only solution of the equation (9).

Case 4. When n is of the form $n = p^\alpha q^\beta$.

Letting $S(n) = S(p^\alpha)$ with $p = 3$, the only possible candidates for (9) to hold true are $n = 2 \cdot 3^\alpha$, $2^2 \cdot 3^\alpha$ where $2 \le \alpha \le 3$. But, $n = 2 \cdot 3^\alpha$ has the divisors $1, 2, 3, 3^2, \cdots, 3^\alpha, 2 \cdot 3, 2 \cdot 3^2, \cdots, 2 \cdot 3^\alpha$, so that

$$\begin{aligned}\sum_{\substack{d \mid n \\ 1 \le d < n}} S(d) &= S(1) + S(2) + S(3) + S(3^2) + \cdots + S(3^\alpha) + S(2 \cdot 3) + S(2 \cdot 3^2) + \cdots + S(2 \cdot 3^\alpha) \\ &= 1 + 2 + 2\sum_{k=1}^{\alpha}(3k) = 1 + 2 + 2 \cdot 3\frac{\alpha(\alpha+1)}{2} = 3(\alpha^2 + \alpha + 1),\end{aligned}$$

and the equation $3(\alpha^2 + \alpha + 1) = 2.3^\alpha$ has no solution for $\alpha \ge 2$.
Again, since $1, 2, 2^2, 3, 3^2, \cdots, 3^\alpha, 2 \cdot 3, 2 \cdot 3^2, \cdots, 2 \cdot 3^\alpha, 2^2 \cdot 3, 2^2 \cdot 3^2, \cdots, 2^2 \cdot 3^\alpha$ are the divisors of $n = 2^2 \cdot 3^\alpha$, we get

$$\begin{aligned}\sum_{\substack{d \mid n \\ 1 \le d < n}} S(d) &= S(1) + S(2) + S(2^2) + S(3) + S(3^2) + \cdots + S(3^\alpha) \\ &+ S(2 \cdot 3) + S(2 \cdot 3^2) + \cdots + S(2 \cdot 3^\alpha) + S(2^2 \cdot 3) + S(2^2 \cdot 3^2) + \cdots + S(2^2 \cdot 3^\alpha) \\ &= 1 + 2 + 4 + 2\sum_{k=1}^{\alpha}(3k) + 4 + \sum_{k=2}^{\alpha}(3k),\end{aligned}$$

which is not divisible by 3.
When $p = 5$, the only possible candidates for (9) to hold true are $n = 5 \cdot 2^2, 2 \cdot 5^2, 2^2 \cdot 5^2$. But, when $n = 2 \cdot 5^2$, the divisors of n are $1, 2, 5, 5^2, 2 \cdot 5$ and $2 \cdot 5^2$, so that

$$\sum_{\substack{d \mid n \\ 1 \le d \le n}} S(d) = S(1) + S(2) + S(5) + S(5^2) + S(2 \cdot 5) + S(2 \cdot 5^2) = 33,$$

and the divisors of $n = 2^2 \cdot 5^2$ are $1,\ 2,\ 2^2\ , 5,\ 5^2,\ 2 \cdot 5,\ 2^2 \cdot 5,\ 2 \cdot 5^2$ and $2^2 \cdot 5^2$ with

$$\begin{aligned}&\sum_{\substack{d \mid n \\ 1 \le d < n}} S(d) = S(1) + S(2) + S(2^2) + S(5) + S(5^2) \\ &+ S(2 \cdot 5) + S(2^2 \cdot 5) + S(2 \cdot 5^2) + S(2^2 \cdot 5^2) = 52.\end{aligned}$$

Thus, there is no solution of (9) of the form $n = p^\alpha q^\beta$.
In the general case when $n = p^\alpha\ q^\beta\ p_1^{\alpha_1}\ p_2^{\alpha_2} \cdots\ p_k^{\alpha_k}$ with $k \geq 1$, an analysis similar to the proof of Theorem 2.1 shows that the equation (9) has no solution.

§3. Remarks

Sandor [2] has considered the problem of finding the S-perfect and completely S-perfect numbers, but his proof is not complete. He has proved that the only S-perfect of the form $n = p\,q$ is $n = 6$ and there is no S-perfect number of the form $n = 2^k q$ where $k \geq 2$ is an integer and q is an odd prime. On the other hand, Theorem 2.1 gives all the S-perfect numbers. Again, Sandor only proved that, the only completely S-perfect number of the form $n = p^2 q$ is $n = 28$, and all completely S-perfect numbers are given by Theorem 2.2.

Theorem 2.1 and Theorem 2.2 find respectively the S-perfect and completely S-perfect numbers when $S(1) = 1$. The situation is quite different if one adopts the convention that $S(0) = 1$. In the latter case, as has been proved by Gronas [3], all completely S-perfect numbers are $n = p$(prime), 9, 16, 24. All that is known about the S-perfect numbers is that, among the first 10^6 numbers, $n = 12$ is the only S-perfect number (see Ashbacher [4]).

In exactly the same way, the Z-perfect and completely Z-perfect numbers may be defined. Thus, given an integer $n \geq 1$,
(1) n is called Z-perfect if and only if

$$n = \sum_{\substack{d|n \\ 1 \leq d \leq n}} Z(d);$$

(2) n is called completely Z-perfect if and only if

$$n = \sum_{\substack{d|n \\ 1 \leq d \leq n}} Z(d);$$

where Z(n) is the pseudo-Smarandache function, defined as follows :

$$Z(n) = \min \left\{ m : m \in \mathbb{Z}^+, n | \frac{m(m+1)}{2} \right\}, n \geq 1.$$

Open Problem. To find all the Z-perfect and completely Z-perfect numbers.

Ashbacher [4] reports, on the basis of computer findings, that the only Z-perfect number less than 10^6 are $n = 4$, 6 (see also Remark 3.5 in Majumdar [5]).

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Cyclic dualizing elements in Girard quantales [1]

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Abstract In this paper, we study the interior structures of Girard quantale and the cyclic dualizing elements of Girard quantale. some equivalent descriptions for Girard quantale are given and an example which shows that the cyclic dualizing element is not unique is given.

Keywords Complete lattice, quantale, Girard quantale, cyclic dualizing element.

§1. Preliminaries

Quantales were introduced by C.J. Mulvey in [1] with the purpose of studying the spectrum of C^*-algebras and the foundations of quantum mechanics. The study of such partially ordered algebraic structures goes back to a series of papers by Ward and Dilworth [2, 3] in the 1930s. It has become a useful tool in studying noncommutative topology, linear logic and C^*-algebra theory [4-6]. Following Mulvey, various types and aspects of quantales have been considered by many researchers [7-9]. The importance of quantales for linear logic is revealed in Yetter's work [10]. Yetter has clarified the use of quantales in linear logic and he has introduced the term "Girard quantale". In [11], J. Paseka and D. Kruml have shown that any quantale can be embedded into a unital quantale. In [12], K.I. Rosenthal has proved that every quantale can be embedded into a Girard quantale. Thus, it is important to study Girard quantale. This is the motivation for us to investigate Girard quantale. In the note, we shall study the interior structures of Girard quantale and the cyclic dualizing element in Girard quantales.

We use 1 to denote the top element and 0 the bottom element in a complete lattice. For notions and concepts, but not explained, please to refer to [12].

Definition 1.1. A quantale is a complete lattice Q with an associative binary operation "&" satisfying:

$$a\&(\bigvee b_\alpha) = \bigvee(a\&b_\alpha) \quad \text{and} \quad (\bigvee b_\alpha)\&a = \bigvee(b_\alpha\&a)$$

for all $a \in Q, \{b_\alpha\} \subseteq Q$.

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An element $e \in Q$ is called a unit if $a\&e = e\&a = a$ for all $a \in Q$. Q is called unital if Q has the unit e.

Since $a\&\_$ and $\_\&a$ preserve arbitrary sups for all $a \in Q$, they have right adjoints and we shall denote them by $a \longrightarrow_r \_$ and $a \longrightarrow_l \_$ respectively.

Proposition 1.2. Let Q be a quantale, $a, b, c \in Q$. Then

(1) $a\&(a \longrightarrow_r b) \leq b$;

(2) $a \longrightarrow_r (b \longrightarrow_r c) = b\&a \longrightarrow_r c$;

Again, analogous results hold upon replacing $\longrightarrow_r$ by $\longrightarrow_l$.

Definition 1.3. Let Q be a quantale, An element c of Q is called cyclic, if $a \longrightarrow_r c = a \longrightarrow_l c$ for all $a \in Q$. $d \in Q$ is called a dualizing element, if $a = (a \longrightarrow_l d) \longrightarrow_r d = (a \longrightarrow_r d) \longrightarrow_l d$ for all $a \in Q$.

Definition 1.4. A quantale Q is called a Girard quantale if it has a cyclic dualizing element d.

Let Q be a Girard quantale with cyclic dualizing element d and $a, b \in Q$, define the binary operation "$\|$ " by $a\|b = (a^{\perp}\&b^{\perp})^{\perp}$, then we can prove that $a\|\_$ and $\_\|a$ preserve arbitrary infs for all $a \in Q$, hence they have left adjoints and we shall denote them by $a \longmapsto_r \_$ and $a \longmapsto_l \_$ respectively. If $a \longrightarrow_r d = a \longrightarrow_l d$, we shall denote it by $a \longrightarrow d$, or more frequently by $a^{\perp}$ if d is a cyclic dualizing element.

§2. The equivalent descriptions for Girard quantale

In this section, we shall study the interior structures of Girard quantale and give some equivalent descriptions for Girard quantale. According to the above, we know that there are six binary operations on a Girard quantale such as $\_\&\_$, $\_ \longrightarrow_r \_$, $\_ \longrightarrow_l \_$, $\_\|\_$, $\_ \longmapsto_r \_$, $\_ \longmapsto_l \_$, we shall respectively call them multiplying, right implication, left implication, Par operation, dual right implication and dual left implication for convenience.

Theorem 2.1. Let Q be a unital quantale, $^{\perp} : Q \longrightarrow Q$ an unary operation on Q. Then Q is a Girard quantale if and only if

(1) $a \longrightarrow_l b = (a\&b^{\perp})^{\perp}$; (2) $a \longrightarrow_r b = (b^{\perp}\&a)^{\perp}$.

Proof. The necessity is obvious. Sufficiency: suppose (1) and (2) hold, $a \in Q$. Denote the unit element by e on Q, then $a = e \longrightarrow_l a = (e\&a^{\perp})^{\perp} = (a^{\perp})^{\perp}$. Thus $a = a^{\perp\perp}$, hence

$$a \longrightarrow_l e^{\perp} = (a\&e^{\perp\perp})^{\perp} = (a\&e)^{\perp} = a^{\perp}.$$

Similarly we get $a^{\perp} = a \longrightarrow_r e^{\perp}$. Take $d = e^{\perp}$, thus d is a cyclic element of Q. Again $\forall a \in Q, (a \longrightarrow e^{\perp}) \longrightarrow e^{\perp} = a^{\perp} \longrightarrow e^{\perp} = (a^{\perp})^{\perp} = a$. This proves $d = e^{\perp}$ is a dualizing element on Q. Thus the proof is completed.

Theorem 2.2. Let Q be a complete lattice. $\_ \longrightarrow_r \_ : Q \times Q \longrightarrow Q$ is a binary operation on Q, $a \longrightarrow_r \_ : Q \longrightarrow Q$ and $\_ \longrightarrow_r a : Q \longrightarrow Q^{op}$ preserve arbitrary sups for all $a \in Q$. $^{\perp} : Q \longrightarrow Q$ is a unary operation on Q, $e \in Q$. For all $a, b, c \in Q$,

(1) $e \longrightarrow_r a = a$; $a \longrightarrow_r e^{\perp} = a^{\perp}$;

(2) $(a^{\perp})^{\perp} = a$; $a \leq b \Longrightarrow b^{\perp} \leq a^{\perp}$;

(3) $(a \longrightarrow_r b^{\perp})^{\perp} \longrightarrow_r c = a \longrightarrow_r (b \longrightarrow_r c)$;

(4) $c \leq a \longrightarrow_r b^{\perp} \Longleftrightarrow a \leq b \longrightarrow_r c^{\perp}$.

Then Q is a Girard quantale and $\_ \longrightarrow_r \_$ is the right implication operation.

Proof. Define the binary operation $a\&b = (b \longrightarrow_r a^{\perp})^{\perp}$ for all $a, b \in Q$.

& satisfies associative law: In fact, for all $a, b, c \in Q$,

$$(a\&b)\&c = (b \longrightarrow_r a^{\perp})^{\perp}\&c = (c \longrightarrow_r (b \longrightarrow_r a^{\perp}))^{\perp}.$$

Using the condition (3), we have $(c \longrightarrow_r (b \longrightarrow_r a^{\perp}))^{\perp} = ((c \longrightarrow_r b^{\perp})^{\perp} \longrightarrow_r a^{\perp})^{\perp}$. By the definition of the binary operation & we can get

$$a\&(b\&c) = a\&(c \longrightarrow_r b^{\perp})^{\perp} = ((c \longrightarrow_r b^{\perp})^{\perp} \longrightarrow_r a^{\perp})^{\perp}.$$

So $(a\&b)\&c = a\&(b\&c)$.

Using the condition (4), we have $a\&b \leq c \Longleftrightarrow (b \longrightarrow_r a^{\perp})^{\perp} \leq c \Longleftrightarrow c^{\perp} \leq b \longrightarrow_r a^{\perp} \Longleftrightarrow b \leq a \longrightarrow_r c$ for all $a, b, c \in Q$.

For any $a \in Q, \{b_i\}_{i\in I} \subseteq Q$. If $I = \emptyset$, then $a\&0 = (a \longrightarrow_r 0^{\perp})^{\perp} = (a \longrightarrow_r 1)^{\perp}$, since again $(a \longrightarrow_r 1)^{\perp} \leq 0 \Longleftrightarrow 1 \leq a \longrightarrow_r 1 \Longleftrightarrow a\&1 \leq 1$, the last inequality obviously holds. So $a\&0 = 0$. Thus $a\&\_$ preserves empty-sups. If $I \neq \emptyset$, then

$$\begin{aligned} a\&(\bigvee_{i\in I} b_i) &= ((\bigvee_{i\in I} b_i) \longrightarrow_r a^{\perp})^{\perp} \\ &= (\bigwedge_{i\in I}(b_i \longrightarrow_r a^{\perp}))^{\perp} \\ &= \bigvee_{i\in I}(b_i \longrightarrow_r a^{\perp})^{\perp} \\ &= \bigvee_{i\in I}(a\&b_i). \end{aligned}$$

Hence $a\&\_$ preserves arbitrary sups for all $a \in Q$. Similarly, we can prove $\_\&a$ preserves arbitrary sups for all $a \in Q$. Thus $(Q, \&)$ is a quantale.

In accordance with the condition (1), we know e is the unit element corresponding to & on Q and $a \longrightarrow_r e^{\perp} = a^{\perp}$. Denote by $a \longrightarrow_l \_$ the right adjoint of $\_\&a$. Then

$$\begin{aligned} a \longrightarrow_l e^{\perp} &= \bigvee\{x \in Q | x \leq a \longrightarrow_l e^{\perp}\} \\ &= \bigvee\{x \in Q | x\&a \leq e^{\perp}\} \\ &= \bigvee\{x \in Q | (a \longrightarrow_r x^{\perp})^{\perp} \leq e^{\perp}\} \\ &= \bigvee\{x \in Q | e \leq a \longrightarrow_r x^{\perp}\} \\ &= \bigvee\{x \in Q | a\&e \leq x^{\perp}\} \\ &= \bigvee\{x \in Q | a \leq x^{\perp}\} \\ &= \bigvee\{x \in Q | x \leq a^{\perp}\} \\ &= a^{\perp}. \end{aligned}$$

This show $e^{\perp}$ is a cyclic element in Q. Using conditions (1) and (2) we know $e^{\perp}$ is also a dualizing element on Q. Hence $(Q, \&, \perp)$ is a Girard quantale. We can easily prove $\_ \longrightarrow_r \_$ is the right implication operation on Q by the above consideration.

Theorem 2.3. Let Q be a complete lattice. $\_ \longmapsto_r \_ : Q \times Q \longrightarrow Q$ is a binary operation in Q, $a \longmapsto_r \_ : Q \longrightarrow Q$ and $\_ \longmapsto_r a : Q^{op} \longrightarrow Q$ preserve arbitrary sups for all $a \in Q$. $^{\perp} : Q \longrightarrow Q$ is an unary operation in Q, $d \in Q$. For all $a, b, c \in Q$,

(1) $d \longmapsto_r a = a$; $a \longmapsto_r d^\perp = a^\perp$;
(2) $(a^\perp)^\perp = a$; $a \leq b \Longrightarrow b^\perp \leq a^\perp$;
(3) $(a \longmapsto_r b)^\perp \longmapsto_r c = a \longmapsto_r (b^\perp \longmapsto_r c)$;
(4) $c \geq a \longmapsto_r b \Longleftrightarrow b^\perp \geq c \longmapsto_r a^\perp$.

Then Q is a Girard quantale and $\_ \longmapsto_r \_$ is the dual right implication operation.

Proof. Define binary operation $a \& b = (b^\perp \longmapsto_r a)$ for all $a, b \in Q$,

(i) The binary operation & is associative: Since $\forall a, b, c \in Q$,

$$\begin{aligned}(a\&b)\&c &= (b^\perp \longmapsto_r a)\&c \\ &= c^\perp \longmapsto_r (b^\perp \longmapsto_r a) \\ &= (c^\perp \longmapsto_r b)^\perp \longmapsto_r a \\ &= a\&(c^\perp \longmapsto_r b) \\ &= a\&(b\&c).\end{aligned}$$

(ii) Using the condition (2), we can prove

$$(\bigvee_{i\in I} a_i)^\perp = \bigwedge_{i\in I}(a_i)^\perp; \quad (\bigwedge_{i\in I} a_i)^\perp = \bigvee_{i\in I}(a_i)^\perp$$

for any set I and $\{a_i\}_{i\in I} \subseteq Q$.

(iii) For all $a \in Q, \{b_i\}_{i\in I} \subseteq Q$, we have

$$a\&(\bigvee_{i\in I} b_i) = (\bigvee_{i\in I} b_i)^\perp \longmapsto_r a = \bigwedge_{i\in I}(b_i)^\perp \longmapsto_r a = \bigvee_{i\in I}(b_i^\perp \longmapsto_r a) = \bigvee_{i\in I}(a\&b_i).$$

Similarly we have $(\bigvee_{i\in I} b_i)\&a = \bigvee_{i\in I}(b_i\&a)$. Hence $(Q,\&)$is a quantale. Since $a\&d^\perp = (d^\perp)^\perp \longmapsto_r a = d \longmapsto_r a = a$; $d^\perp\&b = b^\perp \longmapsto_r d^\perp = b$, thus $(Q,\&)$ is a unit quantale with unit element $d^\perp$.

(iv) If $a \in Q$, we have

$$\begin{aligned}a \longrightarrow_l d &= \bigvee\{x \in Q | x \leq a \longrightarrow_l d\} \\ &= \bigvee\{x \in Q | x\&a \leq d\} \\ &= \bigvee\{x \in Q | a^\perp \longmapsto_r x \leq d\} \\ &= \bigvee\{x \in Q | d \longmapsto_r a \leq x^\perp\} \\ &= \bigvee\{x \in Q | x \leq a^\perp\} \\ &= a^\perp.\end{aligned}$$

Similarly, $a \longrightarrow_r d = a^\perp$, hence d is a cyclic element in Q. d is also a dual element in Q by condition (2). Thus (Q, &) is a Girard quantale with cyclic dual element d. We easily know $\_ \longmapsto_r \_$ is the dual right implication operation in Q by the definition of &.

Obviously, Theorem 2.2 and Theorem 2.3 also hold if $\longrightarrow_r$ and $\longmapsto_r$ are substituted by $\longrightarrow_l$ and $\longmapsto_l$ respectively, "right" and "left" replace each other.

Theorem 2.4. Let Q be a unital quantale with a unary operation $^\perp$ satisfying the condition

$$\textbf{CN}: \quad (a^\perp)^\perp = a \quad \text{and} \quad a \longrightarrow_r b = b^\perp \longrightarrow_l a^\perp$$

for all $a, b \in Q$. Then Q is a Girard quantale.

§3. The cyclic dualizing element of Girard quantale

According to the definition of Girard quantale, we know that the cyclic dualizing element plays an important role in Girard quantale, so we shall discuss the cyclic dualizing element in this section. We shall account for whether the cyclic dualizing element is unique in a Girard quantale; when it is unique; whether these Girard quantales determined by different cyclic dualizing elements are different. Let us see the following example

Example 3.1. Let $Q = \{0, a, b, c, 1\}$, the partial order on Q be defined as Fig 1, the operator & on Q be defined by Table 1. Then we can prove that Q is a commutative Girard quantale. And we can prove that a, b and c are cyclic dualizing elements of Q.

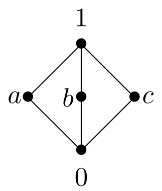

Fig 1

&	0	a	b	c	1
0	0	0	0	0	0
a	0	b	c	a	1
b	0	c	a	b	1
c	0	a	b	c	1
1	0	1	1	1	1

Table 1

Proposition 3.2. Let Q be a unital quantale with the unit element e. $^{\perp_1}$ and $^{\perp_2}$ satisfy the condition **CN** in Theorem 2.4. Then $e^{\perp_1} = e^{\perp_2}$ if and only if $^{\perp_1} = ^{\perp_2}$.

Proposition 3.3. Let Q be a quantale, d_1, d_2 are cyclic dualizing elements of Q, $^{\perp_1}, ^{\perp_2}$ are unary operations on Q induced by d_1, d_2 respectively. Then $d_1 = d_2$ if and only if $^{\perp_1} = ^{\perp_2}$.

Theorem 3.4. Let Q be a Girard quantale. Then there is a one-to-one correspondence between the set of cyclic dualizing elements in Q and the set of unary operations satisfying the condition **CN** in Theorem 2.4.

Proposition 3.5. Let Q be a Girard quantale. If 0 is a cyclic dualizing element of Q, then Q is strictly two-sided.

Proof. Assume 0 is a cyclic dualizing element in Q. Then $0^{\perp} = 0 \longrightarrow 0 = 1$ is the unit of Q, hence $\forall a \in Q, a\&1 = 1\&a = a$, this finished the proof.

Proposition 3.6. If Q is a two-sided Girard quantale, then the unique cyclic dualizing element is the least element 0.

Proof. If Q is a two-sided Girard quantale, then we have $a = a\&e \leq a\&1 \leq a$ for all $a \in Q$. Similarly, we have $1\&a = a$. Thus Q is strictly two-sided. Suppose d is a cyclic dual element in Q, $^{\perp}$ is the unary operation induced by d, then we have $d = 1 \longrightarrow d = 1^{\perp} = 0$. the proof is finished.

Corollary 3.7. Let Q be a Girard quantale with cyclic dualizing element 0. Then the cyclic dualizing element of Q is unique.

Theorem 3.8. Any complete lattice implication algebra is a Girard quantale with unique cyclic dualizing element 0.

According the above conclusions, we have a question : Whether the cyclic dualizing element must be the least element 0 if a Girard quantale has an unique cyclic dualizing element. The

answer is negative. Let us see the following example.

Example 3.9. Let $Q = \{0, e, 1\}$, the partial order on Q be defined by $0 < e < 1$, the binary operation & be defined by Table 2

&	0	e	1
0	0	0	0
e	0	e	1
1	0	1	1

Table 2

It is immediate to verify Q being a Girard quantale with the unique cyclic dualizing element e.

Question 3.10. What is the necessary condition when the cyclic dualizing element of Girard quantale is unique?

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On an equation involving the Smarandache function and the Dirichlet divisor function

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Abstract For any positive integer n, the famous F.Smarandache function $S(n)$ is defined as the smallest positive integer m such that $n \mid m!$. That is to say, $S(n) = \min\{m : n \mid m!, n \in N\}$. $d(n)$ denotes the Dirichlet divisor function. The main purpose of this paper is using the elementary methods to study the solvability of the equation $S(n) = d(n)$, and give its all positive integer solutions.

Keywords The Smarandache function, Dirichlet divisor function, equation, positive integer-solution.

§1. Introduction and result

For any positive integer n, the famous F.Smarandache function $S(n)$ is defined as the smallest positive integer m such that $n \mid m!$. That is to say, $S(n) = \min\{m : n \mid m!, n \in N\}$. From this definition we know that if $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_s^{\alpha_s}$ be the prime powers factorization of n, then $S(n) = \max_{1 \le i \le r} \{S(p_i^{\alpha_i})\}$. About the properties of $S(n)$, many people had studied it, and obtained a series results, see references [4], [5] and [6]. On the other hand, for any positive integer n, the famous Dirichlet divisor function $d(n)$ is defined as the number of all distinct divisors of n. If $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_s^{\alpha_s}$ be the prime power factorization of n, then $d(n) = (\alpha_1 + 1)(\alpha_2 + 1) \cdots (\alpha_s + 1)$. About function $d(n)$, some scholars had studied it, and obtained some deeply conclusions, see references [7] and [8].

In this paper, we shall use the elementary methods to study the solvability of the equation $S(n) = d(n)$, and prove that this equation has infinite positive integer solutions. That is, we shall prove the following main conclusion:

Theorem. For any positive integer n, the equation

$$S(n) = d(n)$$

holds if and only if $n = 2^{2^n - 1}$, $(n = 0,\ 1,\ 2 \cdots)$, and $n = p^\alpha \cdot m$, where $m > 1$ and $m \Bigg| \dfrac{[(\alpha_1 + 1)(\alpha_2 + 1) \cdots (\alpha_s + 1)]!}{p^\alpha}$, if $\alpha \neq 1$, $p \mid \alpha + 1$, $1 < s < \log 2^{\frac{\alpha \cdot p}{\alpha + 1}}$.

§2. Some preliminary lemmas

In this section, we shall give several simple Lemmas which are necessary in the proof of our theorem. They are stated as follows:

Lemma 1. For any prime p, we have the following inequality:

$$(p-1)\cdot\alpha+1\leq S(p^{\alpha})\leq(p-1)\cdot[\alpha+1+\log p^{\alpha}]+1.$$

Proof. (See reference [3]).

Lemma 2. Let a and b are two positive integers, $S=\{1,\ 2,\ 3,\ ....,\ a\}$. Then the number of all positive integers in S which can be divided by b is $\left[\frac{a}{b}\right]$, where $[x]$ denotes the largest integer less than or equal to x.

Proof. (See reference [1]).

Lemma 3. For any prime p, we have:

$$S(p^{k\cdot p^n})=k\left(\phi(p^n)+\frac{1}{k}\right)p.$$

where $1\leq k\leq p$, and $\phi(n)$ denotes the Euler function.

Proof. (See reference [3]).

§3. Proof of the theorem

In this section, we shall complete the proof of our theorem in five cases:

(1). If $n=1$, then $S(1)=1$, $d(1)=1$, so that $S(1)=d(1)$.

(2). If $n=2$, then we can get $S(2)=2$, $d(2)=2$, so that $S(2)=d(2)$.

(3). If $n=p$, where $p>2$, then we have $S(p)=p$, $d(p)=2$, so $S(p)>d(p)$.

(4). If $n=p^{\alpha}$, and $n\neq 2$, when $\alpha\leq p$, $S(p^{\alpha})=\alpha p$, $d(p^{\alpha})=\alpha+1$, so that $S(p^{\alpha})>d(p^{\alpha})$; when $n=p^{\alpha}$, $\alpha>p$, we discuss it in two ways:

(a). If $p\geq 3$ and $\alpha>3$, then from Lemma 1 we can easily get the inequality $S(p^{\alpha})\geq(p-1)\cdot\alpha+1>\alpha+1=d(p^{\alpha})$. That is, $S(p^{\alpha})>d(p^{\alpha})$.

(b). If $p=2$ and $\alpha\geq 2$. It is obvious that $S(2^{\alpha})$ is an even number, and $d(2^{\alpha})=\alpha+1$, that is to say, equality sign holds if and only if α is an odd number. when $\alpha=2^n-1$, then from the definition of $d(n)$ we may immediately get $d(2^{2^n-1})=2^n$. But we also have $S(2^{2^n-1})=2^n$. In fact from Lemma 3 with $k=1$ and $p=2$ we can get

$$S(2^{2^n})=(\phi(2^n)+1)\cdot 2=2^n+2.$$

Since $S(2^{2^n})=2^n+2$, according to the definition of $S(n)$ we have $2^{2^n}\mid(2^n+2)!$, so that $2^{2^n-1}\mid 2^n!(2^{n-1}+1)$, because $2^{n-1}+1$ is an odd number, we can deduce that $2^{2^n-1}\mid 2^n!$ and $2^{2^n}\dagger 2^n!$. So $2^{2^n-1}\dagger(2^n-1)!\cdot 2^{n-1}$, or $2^{2^n-1}\dagger(2^n-2)!$. That is to say, $S(2^{2^n-1})=2^n=d(n)$.

When $\alpha=k\cdot 2^n-1$, $k\geq 3$, and k is an odd number, we have

$$S(2^{k\cdot 2^n-1})-d(2^{k\cdot 2^n-1})=2.$$

In fact from Lemma 1 we can get the inequality

$$S(2^{k\cdot 2^n-1})\geq k\cdot 2^n-1+1=k\cdot 2^n.$$

Especially the equality holds if and only if $k=1$.

If $k \geq 3$ is an odd number, then we have

$$\begin{aligned}\sum_{i=1}^{\infty}\left[\frac{k\cdot 2^n}{2^i}\right] &= k\cdot 2^{n-1}+k\cdot 2^{n-2}+\cdots+\sum_{j=1}^{\infty}\left[\frac{k}{2^j}\right]\\ &= k\cdot 2^n-k+\sum_{j=1}^{\infty}\left[\frac{k}{2^j}\right]. \qquad (1)\end{aligned}$$

It is obvious that

$$k\cdot 2^n-k+\sum_{j=1}^{\infty}\left[\frac{k}{2^j}\right]-k\cdot 2^n+1<0.$$

So $S(2^{k\cdot 2^n-1})>k\cdot 2^n$. That is to say, $2^{k\cdot 2^n-1}\dagger k\cdot 2^n!$.

We also have

$$\begin{aligned}\sum_{i=1}^{\infty}\left[\frac{k\cdot 2^n+2}{2^i}\right] &= k(2^{n-1}+2^{n-2}+\cdots+1)+\left(1+\frac{1}{2}+\cdots+\frac{1}{2^{n-1}}\right)+\sum_{j=n+1}^{\infty}\left[\frac{k\cdot 2^n+2}{2^j}\right]\\ &= k\cdot 2^n-k+2-2\cdot\left(\frac{1}{2}\right)^n+\sum_{j=n+1}^{\infty}\left[\frac{k\cdot 2^n+2}{2^j}\right]. \qquad (2)\end{aligned}$$

It is clear that

$$k\cdot 2^n-k+2-2\cdot\left(\frac{1}{2}\right)^n+\sum_{j=n+1}^{\infty}\left[\frac{k\cdot 2^n+2}{2^j}\right]-k\cdot 2^n+1\geq 0.$$

That means $2^{k\cdot 2^n-1}\mid(k\cdot 2^n+2)!$ and $S(2^{k\cdot 2^n-1})=k\cdot 2^n+2=d(n)+2$.

(5). If $n=p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_s^{\alpha_s}$, then we have

$$d(n)=(\alpha_1+1)(\alpha_2+1)\cdots(\alpha_s+1).$$

$$S(n)=\max_{1\leq i\leq r}\{S(p_i^{\alpha_i})\}=S(p^{\alpha}).$$

Since we have the inequality

$$(\alpha_1+1)(\alpha_2+1)\cdots(\alpha_s+1)-\alpha p\geq 2^{s-1}(\alpha+1)-\alpha p$$

and $S(p^\alpha)\leq\alpha p$, so if $2^{s-1}(\alpha+1)-\alpha p>0$, we can get $s>\log 2^{\frac{\alpha p}{\alpha+1}}$. In this case $S(n)\neq d(n)$.

If $1<s\leq\log 2^{\frac{\alpha p}{\alpha+1}}$, we suppose that

$$S(n)=d(n)=(\alpha_1+1)(\alpha_2+1)\cdots(\alpha_s+1).$$

We can write $n=p^\alpha\cdot m$. $(p^\alpha,\ m)=1$, then $p^\alpha\mid n$. From the definition of $S(n)$ we have $n\mid(\alpha_1+1)(\alpha_2+1)\cdots(\alpha_s+1)!$. So $p^\alpha\cdot m\mid(\alpha_1+1)(\alpha_2+1)\cdots(\alpha_s+1)!$.

Now we discuss it in following three cases:

(i). If $\alpha=1$, then $n=p\cdot m$, $p\neq 2$, then, $S(n)=p$, $d(n)=2d(m)$, so $S(n)\neq d(n)$.

(ii). If $1<\alpha\leq p$, $p\neq 2$, then $n=p^\alpha\cdot m$, so that $S(n)=\alpha\cdot p=(\alpha_1+1)(\alpha_2+1)\cdots(\alpha_s+1)$. If $p\dagger\alpha+1$, then $(\alpha_1+1)(\alpha_2+1)\cdots(\alpha_s+1)\geq(\alpha+1)\cdot p>\alpha\cdot p$, so $S(n)\neq d(n)$. If $p\mid\alpha+1$, we can deduce $\alpha=p-1$, so $n=p^{p-1}\cdot m$, therefore, $m\,\Big|\,\dfrac{((p-1)\cdot p)!}{p^{p-1}}$.

(iii). If $\alpha > p$, then $n = p^{\alpha} \cdot m$, so that $S(n) = t \cdot p = (\alpha_1 + 1)(\alpha_2 + 1) \cdots (\alpha_s + 1) < \alpha \cdot p$. If $p \dagger \alpha + 1$, then $(\alpha_1 + 1)(\alpha_2 + 1) \cdots (\alpha_s + 1) \geq (\alpha + 1) \cdot p \; > \; \alpha \cdot p$, so $S(n) \neq d(n)$. If $p \mid \alpha + 1$, we have $n = p^{\alpha} \cdot m$, therefore, $m \left| \ \frac{[(\alpha_1 + 1)(\alpha_2 + 1) \cdots (\alpha_s + 1)]!}{p^{\alpha}} \right.$.

This completes the proof of our theorem.

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An equation involving the Euler function and the Smarandache m-th power residues function

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Abstract For any positive integer n, let $\phi(n)$ be the Euler function, and $a_m(n)$ denotes the Smarandache m-th power residues function of n. The main purpose of this paper is using the elementary method to study the solvability of the equation $\phi(\phi(n)) = a_m(n)$, and give its all positive integer solutions.

Keywords Smarandache m-th power residues function, Euler function, equation, positive integer solution.

§1. Introduction and result

Let m be a fixed positive integer with $m \geq 2$. For any positive integer n, the Smarandache m-th power residues function $a_m(n)$ is defined as $a_m(1) = 1$, if $n > 1$ and $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ denotes the factorization of n into prime powers, then $a_m(n) = p_1^{\beta_1} p_2^{\beta_2} \cdots p_k^{\beta_k}$, where $\beta_i = \min\{m-1,\ \alpha_i\}$, $i = 1,\ 2,\ \cdots,\ k$. This function was introduced by professor F.Smarandache in his book "Only problems, not solutions ", where he also asked us to study the properties of $a_m(n)$. About this problem and related contents, many authors had studied it, and obtained a series interesting results, see references [1], [2], [3] and [4]. For example, Zhang Tianping [4] studied the value distribution problem of $a_3(n)b_k(n)$, and proved the following conclusion:

$$\sum_{n \leq x} a_3(n) b_k(n) = \frac{6x^{k+1}}{(k+1)\pi^2} \cdot R(k+1) + O\left(x^{k+\frac{1}{2}+\epsilon}\right),$$

where $k \geq 2$ be a positive integer, ϵ denotes any fixed positive number, $b_k(n)$ is the Smarandache k-power complement function,

$$R(k+1) = \prod_p \left(1 + \frac{p^3+p}{p^7+p^6-p-1}\right),\ k = 2$$

and

$$R(k+1) = \prod_p \left(1 + \sum_{j=2}^{k} \frac{p^{k-j+3}}{(p+1)p^{(k+1)j}} + \sum_{j=1}^{k} \frac{p^{k-j+3}}{(p+1)(p^{(k+1)(k+j)} - p^{(k+1)j})}\right),\ k > 2.$$

In this paper, we shall study the solvability of the equation

$$\phi(\phi(n)) = a_m(n), \tag{1}$$

where $\phi(n)$ is the famous Euler function. About this content, Zhang Wenpeng [5] studied the solvability of the equation $\phi(n) = a_m(n)$, and obtained its all positive integer solution. But for equation (1), it seems that none had studied it yet, at least we have not seen any related papers before. In this paper, we use the elementary method to obtain all positive integer solutions of the equation (1). That is, we shall prove the following conclusion:

Theorem. Let m be a fixed positive integer with $m \geq 2$. Then the equation (1) have $m+1$ positive integer solutions, they are:

$$n = 1,\ 2^{m+1},\ 2^{\alpha}\cdot 3^{m+1},\ \ \alpha = 1,\ 2,\ 3,\ \cdots,\ m-1.$$

It is clear that using the method of proving our Theorem we can also obtain all positive integer solutions of the equation $\phi(\phi(\phi(n))) = a_m(n)$.

§2. Proof of the theorem

In this section, we shall use the elementary method to complete the proof of our Theorem directly. About the properties of the Euler function, it can be found in references [8] and [9]. It is clear that $n = 1$ is a solution of (1). Now we assume that $n > 1$, let $n = p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k}$ denotes the factorization of n into prime powers, then from the definition of $\phi(n)$ and $a_m(n)$ we have

$$a_m(n) = p_1^{\beta_1}p_2^{\beta_2}\cdots p_k^{\beta_k},\ \ \beta_i = \min\{m-1,\ \alpha_i\},\ \ i = 1,\ 2,\ \cdots,\ k \tag{2}$$

and

$$\phi(n) = p_1^{\alpha_1-1}(p_1-1)p_2^{\alpha_2-1}(p_2-1)\cdots p_k^{\alpha_k-1}(p_k-1). \tag{3}$$

Now we discuss the problem in following four cases:

(A) If $\exists\ \alpha_i > m+1$, then $p_i^{m+1} \mid \phi(n)$, and $p_i^m \mid \phi(\phi(n))$. But from (2) we know that all $\beta_i \leq m-1$, so $\phi(\phi(n)) \neq a_m(n)$. That is, the equation (1) does not hold.

(B) If $\alpha_1 = \alpha_2 = \cdots = \alpha_k = m+1$, then from (2) and (3) we have

$$a_m(n) = p_1^{m-1}p_2^{m-1}p_3^{m-1}\cdots p_k^{m-1}$$

and

$$\phi(n) = p_1^m(p_1-1)p_2^m(p_2-1)p_3^m(p_3-1)\cdots p_k^m(p_k-1).$$

If p_i divide one of $(p_{i+j}-1)$, where $(i = 1,\ 2,\ \cdots,\ k-1;\ j = 1,\ 2,\ \cdots, k-i)$, then $p_i^m \mid \phi(\phi(n))$, so that $\phi(\phi(n)) \neq a_m(n)$. Note that if $\phi(\phi(n)) = a_m(n)$, then $\phi(\phi(n))$ and $a_m(n)$ have the same prime divisors. So in this case, the equation (1) holds only if $n = 2^{m+1}$.

(C) If $\max\limits_{1\leq i\leq k}\{\alpha_i\} < m+1$, then $p_k^{\alpha_k-1} \parallel \phi(n)$ and $p_k^{\alpha_k-2} \parallel \phi(\phi(n))$, but $\alpha_k - 2 < m-1$ and $\beta_k = \min\{m-1,\ \alpha_k\} > \alpha_k - 2$. So in this case, the equation (1) has no solution.

(D) If $\exists\ i,\ j$ such that $\alpha_i = m+1,\ \alpha_j < m+1$. In this time, if n satisfy the equation (1), then n must be an even number, and $p_1 = 2$. In this case, we can easily deduce that $k \leq 2$. In fact if $k \geq 3$, then note that $p_1 = 2,\ 2 \mid p_2 - 1,\ 4 \mid p_3 - 1$ or $2 \mid \phi\left(\frac{p_3-1}{2}\right)$, from $\phi(n) = p_1^{\alpha_1-1}(p_1-1)p_2^{\alpha_2-1}(p_2-1)\cdots p_k^{\alpha_k-1}(p_k-1)$ we may immediately deduce that $2^{\alpha_1+1} \mid \phi(n)$, but $2^{\alpha_1+1} \dagger a_m(n)$, so $\phi(\phi(n)) \neq a_m(n)$. So if n satisfy the equation (1), then $k \leq 2$. Since $\exists\ i,\ j$ such that $\alpha_i = m+1,\ \alpha_j < m+1$. So we have $k = 2$ and $n = 2^\alpha p^{m+1}$ or $n = 2^{m+1}p^\alpha$, where $\alpha < m+1$. If $n = 2^\alpha p^{m+1}$, then $\phi(n) = 2^\alpha \cdot \frac{p-1}{2} \cdot p^m$ and $\phi(\phi(n)) = \phi\left(2^\alpha \cdot \frac{p-1}{2}\right) \cdot p^{m-1} \cdot (p-1)$. But $a_m(n) = 2^\beta p^{m-1}$, where $\beta = \min\{m-1,\ \alpha\}$. So $\phi(\phi(n)) = a_m(n)$ if and only if $p - 1 = 2$, or $p = 3$, $\alpha \leq m-1$. That is to say, $n = 2^\alpha \cdot 3^{m+1}$ satisfy the equation (1) for all $\alpha \leq m-1$. If $n = 2^{m+1}p^\alpha$, then 2^{m+1} divide $\phi(\phi(n))$, but $2^{m+1} \dagger a_m(n)$, so this time, n does not satisfy the equation (1).

Combining (A), (B), (C) and (D) we may immediately deduce that all positive integer solutions of the equation (1) are $n = 1,\ 2^{m+1},\ 2^\alpha \cdot 3^{m+1}$, where $\alpha = 1,\ 2,\ \cdots,\ m-1$. This completes the proof of Theorem.

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On numerical values of $Z(n)$

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Abstract The pseudo Smarandache function, $Z(n)$, is defined as the minimum positive integer m such that $1+2+\cdots+m$ is divisible by n. The main purpose of this paper is to reproduce the values of $Z(n)$ for $n = 1$ (1)5000.

Keywords Pseudo Smarandache function, unsolved problems.

§1. Introduction

The pseudo Smarandache function, denoted by $Z(n)$, has been defined by Kashihara [1] as follows.

Definition 1.1. For any integer $n \geq 1$, the pseudo Smarandache function, $Z(n)$ is the minimum integer m such that $1+2+\cdots+m$ is divisible by n, that is

$$Z(n) = \min\left\{m : m \geq 1, n|\frac{m(m+1)}{2}\right\}, n \geq 1.$$

Soon after introduction, the pseudo Smarandache function has attracted the attention of many researchers. However, till date, only little is known about the function. The following lemma, due to Kashihara [1], Ibstedt [2] and Ashbacher [3], summarizes the properties of $Z(n)$ known so far.

Lemma 1.1. For any integer $n \geq 1$,

(1) $1 \leq Z(n) \leq 2n-1$,

(2) $Z(n) \leq n-1$ for any odd integer $n \geq 3$,

with

(a) $Z(n) = 1$ if and only if $n = 1$,

(b) $Z(n) = 2$ if and only if $n = 3$,

(3) $Z(n) \geq \max\{Z(d) : d|n\}$,

(4) $Z(n) \geq \frac{1}{2}(\sqrt{1+8n}-1)$.

As for the expressions for $Z(n)$, we have the following results, due to Kashihara [1] and Ashbacher [3].

Lemma 1.2. For any integer $k \geq 1$,

(1) $Z(\frac{k(k+1)}{2}) = k$,

(2) $Z(2^k) = 2^{k+1}-1$,

(3) $Z(p^k) = p^k - 1$.

In addition to the above expressions, we have explicit expressions for $Z(3\cdot 2^k)$ and $Z(5\cdot 2^k)$, due to Ashbacher [2], and $Z(7\cdot 2^k)$, due to Ibstedt [4]. Majumdar [5] gives explicit expressions for $Z(4p)$, $Z(5p)$, $Z(6p)$, $Z(7p)$, $Z(11p)$ and $Z(pq)$.

On the other hand, questions and conjectures involving $Z(n)$ are quite plenty, some of which have been settled but many still remain unsolved. Some of the unsolved problems are given below.

Problem 1. Given the integers k and m, find all integers n such that $Z^k(n) = m$, (where $Z^k(\cdot)$ denotes the k-fold composition of z with itself).

Problem 2. Find relationships between each of $Z(m+n)$ and $Z(mn)$ with $Z(m)$ and $Z(n)$.

Problem 3. Find all values of n such that

(1) $Z(n+1) = Z(n)$, **(2)** $Z(n)$ divides $Z(n+1)$, **(3)** $Z(n+1)$ divides $Z(n)$.

Problem 4. Are there solutions to

(1) $Z(n+2) = Z(n+1) + Z(n)$?

(2) $Z(n) = Z(n+1) + Z(n+2)$?

(3) $Z(n+2) = Z(n+1)Z(n)$?

(4) $Z(n) = Z(n+1)Z(n+2)$?

(5) $Z(n+2) + Z(n) = 2Z(n+1)$?

(6) $Z(n+2)Z(n) = [Z(n+1)]^2$?

Problem 5. For a given m, how many n are there such that $Z(n) = m$? Moreover, if $Z(n) = m$ has only one solution, what are the conditions on m?

Problem 6. Are there infinitely many instances of 3 consecutive increasing or decreasing terms in the sequence $\{Z(n)\}_{n=1}^{\infty}$?

Problem 7. Find all solutions of **(1)** $Z(n) = S(n)$, **(2)** $Z(n)+1 = S(n)$.

Recently, Wengpeng Zhang and Ling Li [7] have made the following conjecture :

Conjecture 1. For any integer $n \geq 1$, the equation $Z(n)+S_c(n) = 2n$ has the only solutions $n = 1$, 3^k, p^{2m-1}, where $k \geq 2$ is an integer such that 3^k and $3^k + 2$ are twin primes, and $p \geq 5$ is a prime and $m \geq 1$ is an integer such that p^{2m-1} and $p^{2m-1} + 2$ are twin primes.

Recall that $S_c(n)$ is the Smarandache reciprocal function, $S_c(n)$, is defined as follows : $S_c(n) = x$ if and only if $x + 1$ is the smallest prime greater than n.

To address such problems, it seems that the existing expressions of $Z(n)$ are not sufficient. With this in mind, we give a table of values of $Z(n)$ for $n = 1(1)5000$. It may be mentioned here that, Ibstedt [6] provides a table of values of $S(n)$ over the same range, and our table would supplement that.

§2. Numerical values of $Z(n)$

The values of $Z(n)$ have been calculated on a computer, using Definition 1.1. In the tables that follow, we reproduce these values.

Table 1: **Values of Z(n) for n=1(1)273**

n	Z(n)	n	Z(n)	n	Z(n)	n	Z(n)	n	Z(n)	n	Z(n)	n	Z(n)
1	1	40	15	79	78	118	59	157	156	196	48	235	94
2	3	41	40	80	64	119	34	158	79	197	196	236	176
3	2	42	20	81	80	120	15	159	53	198	44	237	78
4	7	43	42	82	40	121	120	160	64	199	198	238	84
5	4	44	32	83	82	122	60	161	69	200	175	239	238
6	3	45	9	84	48	123	41	162	80	201	66	240	95
7	6	46	23	85	34	124	31	163	162	202	100	241	240
8	15	47	46	86	43	125	124	164	40	203	28	242	120
9	8	48	32	87	29	126	27	165	44	204	119	243	242
10	4	49	48	88	32	127	126	166	83	205	40	244	183
11	10	50	24	89	88	128	255	167	166	206	103	245	49
12	8	51	17	90	35	129	42	168	48	207	45	246	123
13	12	52	39	91	13	130	39	169	168	208	64	247	38
14	7	53	52	92	23	131	130	170	84	209	76	248	31
15	5	54	27	93	30	132	32	171	18	210	20	249	83
16	31	55	10	94	47	133	56	172	128	211	210	250	124
17	16	56	48	95	19	134	67	173	172	212	159	251	250
18	8	57	18	96	63	135	54	174	87	213	71	252	63
19	18	58	28	97	96	136	16	175	49	214	107	253	22
20	15	59	58	98	48	137	136	176	32	215	85	254	127
21	6	60	15	99	44	138	23	177	59	216	80	255	50
22	11	61	60	100	24	139	138	178	88	217	62	256	511
23	22	62	31	101	100	140	55	179	178	218	108	257	256
24	15	63	27	102	51	141	47	180	80	219	72	258	128
25	24	64	127	103	102	142	71	181	180	220	55	259	111
26	12	65	25	104	64	143	65	182	91	221	51	260	39
27	26	66	11	105	14	144	63	183	60	222	36	261	116
28	7	67	66	106	52	145	29	184	160	223	222	262	131
29	28	68	16	107	106	146	72	185	74	224	63	263	262
30	15	69	23	108	80	147	48	186	92	225	99	264	32
31	30	70	20	109	108	148	111	187	33	226	112	265	105
32	63	71	70	110	44	149	148	188	47	227	226	266	56
33	11	72	63	111	36	150	24	189	27	228	56	267	89
34	16	73	72	112	63	151	150	190	19	229	228	268	200
35	14	74	36	113	112	152	95	191	190	230	115	269	268
36	8	75	24	114	56	153	17	192	128	231	21	270	80
37	36	76	56	115	45	154	55	193	192	232	144	271	270
38	19	77	21	116	87	155	30	194	96	233	232	272	255
39	12	78	12	117	26	156	39	195	39	234	116	273	77

Table 2: **Values of Z(n) for n=274(1)546**

n	Z(n)	n	Z(n)	n	Z(n)	n	Z(n)	n	Z(n)	n	Z(n)	n	Z(n)
274	136	313	312	352	319	391	68	430	215	469	133	508	127
275	99	314	156	353	352	392	48	431	430	470	140	509	508
276	23	315	35	354	59	393	131	432	351	471	156	510	84
277	276	316	79	355	70	394	196	433	432	472	176	511	146
278	139	317	316	356	88	395	79	434	216	473	43	512	1023
279	62	318	159	357	84	396	143	435	29	474	236	513	189
280	160	319	87	358	179	397	396	436	327	475	75	514	256
281	280	320	255	359	358	398	199	437	114	476	119	515	205
282	47	321	107	360	80	399	56	438	72	477	53	516	128
283	282	322	91	361	360	400	224	439	438	478	239	517	187
284	71	323	152	362	180	401	400	440	175	479	478	518	111
285	75	324	80	363	120	402	200	441	98	480	255	519	173
286	143	325	25	364	104	403	155	442	51	481	221	520	64
287	41	326	163	365	145	404	303	443	442	482	240	521	520
288	63	327	108	366	60	405	80	444	111	483	69	522	116
289	288	328	287	367	366	406	28	445	89	484	120	523	522
290	115	329	140	368	160	407	110	446	223	485	194	524	392
291	96	330	44	369	81	408	255	447	149	486	243	525	125
292	72	331	330	370	184	409	408	448	384	487	486	526	263
293	292	332	248	371	105	410	40	449	448	488	304	527	186
294	48	333	36	372	216	411	137	450	99	489	162	528	32
295	59	334	167	373	372	412	103	451	164	490	195	529	528
296	111	335	134	374	187	413	118	452	112	491	490	530	159
297	54	336	63	375	125	414	207	453	150	492	287	531	117
298	148	337	336	376	47	415	165	454	227	493	203	532	56
299	91	338	168	377	116	416	64	455	90	494	208	533	246
300	24	339	113	378	27	417	138	456	95	495	44	534	267
301	42	340	119	379	378	418	76	457	456	496	31	535	214
302	151	341	154	380	95	419	418	458	228	497	70	536	335
303	101	342	152	381	126	420	104	459	135	498	83	537	179
304	95	343	342	382	191	421	420	460	160	499	498	538	268
305	60	344	128	383	382	422	211	461	460	500	375	539	98
306	135	345	45	384	255	423	188	462	132	501	167	540	80
307	306	346	172	385	55	424	159	463	462	502	251	541	540
308	55	347	346	386	192	425	50	464	319	503	502	542	271
309	102	348	87	387	171	426	71	465	30	504	63	543	180
310	124	349	348	388	96	427	182	466	232	505	100	544	255
311	310	350	175	389	388	428	320	467	466	506	252	545	109
312	143	351	26	390	39	429	65	468	143	507	168	546	104

Table 3: **Values of Z(n) for n=547(1)819**

n	Z(n)	n	Z(n)	n	Z(n)	n	Z(n)	n	Z(n)	n	Z(n)	n	Z(n)
547	546	586	292	625	624	664	415	703	37	742	371	781	142
548	136	587	586	626	312	665	189	704	384	743	742	782	68
549	243	588	48	627	132	666	36	705	140	744	464	783	377
550	99	589	247	628	471	667	115	706	352	745	149	784	735
551	57	590	59	629	221	668	167	707	202	746	372	785	314
552	207	591	197	630	35	669	222	708	176	747	332	786	131
553	237	592	480	631	630	670	200	709	708	748	407	787	786
554	276	593	592	632	79	671	121	710	284	749	321	788	591
555	74	594	296	633	210	672	63	711	315	750	375	789	263
556	416	595	34	634	316	673	672	712	623	751	750	790	79
557	556	596	447	635	254	674	336	713	92	752	704	791	112
558	216	597	198	636	159	675	324	714	84	753	251	792	143
559	129	598	91	637	195	676	168	715	65	754	116	793	182
560	160	599	598	638	87	677	676	716	536	755	150	794	396
561	33	600	224	639	71	678	339	717	239	756	216	795	105
562	280	601	600	640	255	679	97	718	359	757	756	796	199
563	562	602	300	641	640	680	255	719	718	758	379	797	796
564	47	603	134	642	107	681	227	720	224	759	230	798	56
565	225	604	151	643	642	682	340	721	308	760	95	799	187
566	283	605	120	644	160	683	682	722	360	761	760	800	575
567	161	606	303	645	129	684	152	723	240	762	380	801	89
568	496	607	606	646	152	685	274	724	543	763	217	802	400
569	568	608	512	647	646	686	343	725	174	764	191	803	219
570	75	609	174	648	80	687	228	726	120	765	135	804	200
571	570	610	60	649	176	688	128	727	726	766	383	805	69
572	143	611	234	650	299	689	52	728	272	767	117	806	155
573	191	612	135	651	62	690	275	729	728	768	512	807	269
574	287	613	612	652	488	691	690	730	219	769	768	808	303
575	275	614	307	653	652	692	519	731	85	770	55	809	808
576	512	615	164	654	108	693	98	732	183	771	257	810	80
577	576	616	175	655	130	694	347	733	732	772	192	811	810
578	288	617	616	656	287	695	139	734	367	773	772	812	231
579	192	618	308	657	72	696	144	735	195	774	171	813	270
580	144	619	618	658	140	697	204	736	575	775	124	814	296
581	83	620	279	659	658	698	348	737	66	776	96	815	325
582	96	621	161	660	120	699	233	738	287	777	111	816	255
583	264	622	311	661	660	700	175	739	738	778	388	817	171
584	511	623	266	662	331	701	700	740	184	779	246	818	408
585	90	624	351	663	51	702	324	741	38	780	39	819	90

Table 4: **Values of Z(n) for n=820(1)1053**

n	Z(n)	n	Z(n)	n	Z(n)	n	Z(n)	n	Z(n)	n	Z(n)
820	40	859	858	898	448	937	936	976	671	1015	174
821	820	860	215	899	434	938	335	977	976	1016	127
822	411	861	41	900	224	939	312	978	488	1017	225
823	822	862	431	901	424	940	375	979	88	1018	508
824	720	863	862	902	164	941	940	980	440	1019	1018
825	99	864	512	903	42	942	156	981	108	1020	119
826	412	865	345	904	112	943	368	982	491	1021	1020
827	826	866	432	905	180	944	767	983	982	1022	364
828	207	867	288	906	452	945	189	984	287	1023	186
829	828	868	216	907	906	946	43	985	394	1024	2047
830	415	869	395	908	680	947	946	986	203	1025	450
831	276	870	144	909	404	948	552	987	140	1026	323
832	767	871	402	910	104	949	364	988	208	1027	78
833	391	872	544	911	910	950	75	989	344	1028	256
834	416	873	387	912	95	951	317	990	44	1029	342
835	334	874	436	913	165	952	272	991	990	1030	515
836	208	875	125	914	456	953	952	992	960	1031	1030
837	216	876	72	915	60	954	423	993	330	1032	128
838	419	877	876	916	687	955	190	994	496	1033	1032
839	838	878	439	917	392	956	239	995	199	1034	187
840	224	879	293	918	135	957	87	996	248	1035	45
841	840	880	319	919	918	958	479	997	996	1036	111
842	420	881	880	920	160	959	273	998	499	1037	305
843	281	882	440	921	306	960	255	999	296	1038	519
844	632	883	882	922	460	961	960	1000	624	1039	1038
845	169	884	272	923	142	962	259	1001	77	1040	64
846	188	885	59	924	231	963	107	1002	167	1041	347
847	363	886	443	925	74	964	240	1003	118	1042	520
848	159	887	886	926	463	965	385	1004	752	1043	447
849	282	888	111	927	206	966	252	1005	134	1044	144
850	424	889	126	928	319	967	966	1006	503	1045	209
851	184	890	355	929	928	968	847	1007	265	1046	523
852	71	891	242	930	155	969	152	1008	63	1047	348
853	852	892	223	931	342	970	484	1009	1008	1048	655
854	244	893	94	932	232	971	970	1010	100	1049	1048
855	170	894	447	933	311	972	728	1011	336	1050	224
856	320	895	179	934	467	973	139	1012	528	1051	1050
857	856	896	511	935	220	974	487	1013	1012	1052	263
858	143	897	207	936	143	975	299	1014	168	1053	324

Table 5: **Values of Z(n) for n=1054(1)1287**

n	Z(n)	n	Z(n)	n	Z(n)	n	Z(n)	n	Z(n)	n	Z(n)
1054	340	1093	1092	1132	848	1171	1170	1210	120	1249	1248
1055	210	1094	547	1133	308	1172	879	1211	518	1250	624
1056	384	1095	219	1134	567	1173	68	1212	303	1251	278
1057	301	1096	959	1135	454	1174	587	1213	1212	1252	312
1058	528	1097	1096	1136	639	1175	375	1214	607	1253	357
1059	353	1098	243	1137	378	1176	48	1215	485	1254	132
1060	159	1099	314	1138	568	1177	428	1216	512	1255	250
1061	1060	1100	175	1139	67	1178	247	1217	1216	1256	784
1062	531	1101	366	1140	95	1179	261	1218	203	1257	419
1063	1062	1102	551	1141	489	1180	295	1219	529	1258	407
1064	399	1103	1102	1142	571	1181	1180	1220	304	1259	1258
1065	284	1104	575	1143	126	1182	591	1221	110	1260	224
1066	532	1105	169	1144	143	1183	168	1222	376	1261	194
1067	484	1106	315	1145	229	1184	703	1223	1222	1262	631
1068	623	1107	81	1146	191	1185	315	1224	288	1263	420
1069	1068	1108	831	1147	185	1186	592	1225	49	1264	1184
1070	320	1109	1108	1148	287	1187	1186	1226	612	1265	230
1071	153	1110	444	1149	383	1188	296	1227	408	1266	632
1072	736	1111	505	1150	275	1189	492	1228	920	1267	181
1073	406	1112	416	1151	1150	1190	84	1229	1228	1268	951
1074	179	1113	105	1152	512	1191	396	1230	164	1269	188
1075	300	1114	556	1153	1152	1192	447	1231	1230	1270	380
1076	807	1115	445	1154	576	1193	1192	1232	384	1271	123
1077	359	1116	216	1155	209	1194	596	1233	548	1272	159
1078	440	1117	1116	1156	288	1195	239	1234	616	1273	133
1079	415	1118	559	1157	533	1196	207	1235	285	1274	195
1080	80	1119	372	1158	192	1197	189	1236	720	1275	50
1081	46	1120	384	1159	304	1198	599	1237	1236	1276	87
1082	540	1121	531	1160	144	1199	109	1238	619	1277	1276
1083	360	1122	407	1161	215	1200	224	1239	294	1278	71
1084	271	1123	1122	1162	83	1201	1200	1240	464	1279	1278
1085	154	1124	280	1163	1162	1202	600	1241	510	1280	1024
1086	180	1125	125	1164	96	1203	401	1242	459	1281	182
1087	1086	1126	563	1165	465	1204	343	1243	451	1282	640
1088	255	1127	391	1166	264	1205	240	1244	311	1283	1282
1089	242	1128	47	1167	389	1206	468	1245	165	1284	320
1090	435	1129	1128	1168	511	1207	425	1246	356	1285	514
1091	1090	1130	339	1169	167	1208	1056	1247	86	1286	643
1092	104	1131	116	1170	260	1209	155	1248	767	1287	143

Table 6: **Values of Z(n) for n=1288(1)1521**

n	Z(n)	n	Z(n)	n	Z(n)	n	Z(n)	n	Z(n)	n	Z(n)
1288	160	1327	1326	1366	683	1405	280	1444	360	1483	1482
1289	1288	1328	415	1367	1366	1406	703	1445	289	1484	847
1290	215	1329	443	1368	512	1407	335	1446	240	1485	54
1291	1290	1330	399	1369	1368	1408	1023	1447	1446	1486	743
1292	152	1331	1330	1370	684	1409	1408	1448	543	1487	1486
1293	431	1332	296	1371	456	1410	140	1449	161	1488	960
1294	647	1333	558	1372	343	1411	663	1450	724	1489	1488
1295	259	1334	115	1373	1372	1412	352	1451	1450	1490	595
1296	1215	1335	89	1374	228	1413	314	1452	120	1491	426
1297	1296	1336	1168	1375	374	1414	504	1453	1452	1492	1119
1298	176	1337	573	1376	128	1415	565	1454	727	1493	1492
1299	432	1338	668	1377	323	1416	176	1455	194	1494	332
1300	624	1339	103	1378	52	1417	545	1456	832	1495	299
1301	1300	1340	200	1379	196	1418	708	1457	93	1496	527
1302	216	1341	297	1380	344	1419	429	1458	728	1497	498
1303	1302	1342	671	1381	1380	1420	639	1459	1458	1498	427
1304	815	1343	237	1382	691	1421	637	1460	584	1499	1498
1305	144	1344	384	1383	461	1422	315	1461	486	1500	375
1306	652	1345	269	1384	864	1423	1422	1462	731	1501	474
1307	1306	1346	672	1385	554	1424	800	1463	76	1502	751
1308	327	1347	449	1386	252	1425	75	1464	671	1503	333
1309	153	1348	336	1387	437	1426	92	1465	585	1504	704
1310	524	1349	284	1388	1040	1427	1426	1466	732	1505	300
1311	114	1350	324	1389	462	1428	119	1467	162	1506	251
1312	1024	1351	385	1390	139	1429	1428	1468	367	1507	274
1313	403	1352	1183	1391	428	1430	220	1469	338	1508	376
1314	72	1353	164	1392	608	1431	53	1470	195	1509	503
1315	525	1354	676	1393	398	1432	895	1471	1470	1510	604
1316	328	1355	270	1394	204	1433	1432	1472	896	1511	1510
1317	438	1356	791	1395	279	1434	239	1473	491	1512	783
1318	659	1357	413	1396	1047	1435	245	1474	736	1513	356
1319	1318	1358	679	1397	253	1436	359	1475	649	1514	756
1320	384	1359	603	1398	699	1437	479	1476	287	1515	404
1321	1320	1360	255	1399	1398	1438	719	1477	210	1516	1136
1322	660	1361	1360	1400	175	1439	1438	1478	739	1517	369
1323	539	1362	227	1401	467	1440	575	1479	203	1518	252
1324	992	1363	376	1402	700	1441	131	1480	480	1519	588
1325	424	1364	495	1403	183	1442	308	1481	1480	1520	95
1326	51	1365	90	1404	351	1443	221	1482	455	1521	675

Table 7: **Values of Z(n) for n=1522(1)1755**

n	Z(n)	n	Z(n)	n	Z(n)	n	Z(n)	n	Z(n)	n	Z(n)
1522	760	1561	223	1600	1024	1639	297	1678	839	1717	101
1523	1522	1562	780	1601	1600	1640	655	1679	437	1718	859
1524	888	1563	521	1602	711	1641	546	1680	224	1719	764
1525	549	1564	391	1603	686	1642	820	1681	1680	1720	559
1526	763	1565	625	1604	400	1643	371	1682	840	1721	1720
1527	509	1566	783	1605	320	1644	959	1683	153	1722	287
1528	191	1567	1566	1606	219	1645	140	1684	1263	1723	1722
1529	417	1568	832	1607	1606	1646	823	1685	674	1724	431
1530	135	1569	522	1608	335	1647	243	1686	843	1725	275
1531	1530	1570	784	1609	1608	1648	927	1687	482	1726	863
1532	383	1571	1570	1610	160	1649	679	1688	1055	1727	627
1533	146	1572	392	1611	179	1650	99	1689	563	1728	512
1534	767	1573	363	1612	247	1651	507	1690	675	1729	455
1535	614	1574	787	1613	1612	1652	944	1691	266	1730	519
1536	1023	1575	125	1614	807	1653	57	1692	423	1731	576
1537	318	1576	591	1615	170	1654	827	1693	1692	1732	432
1538	768	1577	664	1616	1312	1655	330	1694	363	1733	1732
1539	323	1578	263	1617	98	1656	207	1695	225	1734	288
1540	55	1579	1578	1618	808	1657	1656	1696	1536	1735	694
1541	736	1580	79	1619	1618	1658	828	1697	1696	1736	496
1542	771	1581	186	1620	80	1659	237	1698	848	1737	386
1543	1542	1582	112	1621	1620	1660	415	1699	1698	1738	395
1544	192	1583	1582	1622	811	1661	604	1700	424	1739	517
1545	309	1584	351	1623	540	1662	276	1701	728	1740	144
1546	772	1585	634	1624	608	1663	1662	1702	184	1741	1740
1547	272	1586	792	1625	624	1664	767	1703	130	1742	468
1548	215	1587	528	1626	812	1665	369	1704	639	1743	83
1549	1548	1588	1191	1627	1626	1666	391	1705	154	1744	544
1550	124	1589	454	1628	296	1667	1666	1706	852	1745	349
1551	329	1590	159	1629	180	1668	416	1707	569	1746	387
1552	96	1591	258	1630	815	1669	1668	1708	671	1747	1746
1553	1552	1592	1392	1631	699	1670	500	1709	1708	1748	551
1554	111	1593	648	1632	255	1671	557	1710	360	1749	264
1555	310	1594	796	1633	781	1672	208	1711	58	1750	875
1556	1167	1595	319	1634	171	1673	238	1712	320	1751	102
1557	692	1596	56	1635	435	1674	216	1713	570	1752	656
1558	532	1597	1596	1636	408	1675	200	1714	856	1753	1752
1559	1558	1598	187	1637	1636	1676	1256	1715	685	1754	876
1560	480	1599	246	1638	468	1677	129	1716	143	1755	324

Table 8: **Values of Z(n) for n=1756(1)1989**

n	Z(n)	n	Z(n)	n	Z(n)	n	Z(n)	n	Z(n)	n	Z(n)
1756	439	1795	359	1834	392	1873	1872	1912	239	1951	1950
1757	251	1796	448	1835	734	1874	936	1913	1912	1952	1280
1758	879	1797	599	1836	135	1875	624	1914	87	1953	62
1759	1758	1798	464	1837	835	1876	335	1915	765	1954	976
1760	319	1799	770	1838	919	1877	1876	1916	479	1955	390
1761	587	1800	224	1839	612	1878	312	1917	567	1956	488
1762	880	1801	1800	1840	160	1879	1878	1918	959	1957	721
1763	860	1802	424	1841	525	1880	704	1919	303	1958	88
1764	440	1803	600	1842	920	1881	341	1920	255	1959	653
1765	705	1804	615	1843	873	1882	940	1921	339	1960	735
1766	883	1805	360	1844	1383	1883	538	1922	960	1961	370
1767	341	1806	300	1845	369	1884	471	1923	641	1962	108
1768	272	1807	416	1846	780	1885	260	1924	480	1963	754
1769	609	1808	1695	1847	1846	1886	368	1925	175	1964	1472
1770	59	1809	134	1848	384	1887	221	1926	107	1965	524
1771	230	1810	180	1849	1848	1888	767	1927	328	1966	983
1772	1328	1811	1810	1850	924	1889	1888	1928	240	1967	280
1773	197	1812	1056	1851	617	1890	539	1929	642	1968	287
1774	887	1813	147	1852	463	1891	61	1930	579	1969	715
1775	425	1814	907	1853	544	1892	472	1931	1930	1970	984
1776	480	1815	120	1854	720	1893	630	1932	552	1971	729
1777	1776	1816	1135	1855	105	1894	947	1933	1932	1972	696
1778	888	1817	552	1856	1536	1895	379	1934	967	1973	1972
1779	593	1818	404	1857	618	1896	1184	1935	215	1974	140
1780	800	1819	748	1858	928	1897	812	1936	1088	1975	474
1781	273	1820	104	1859	506	1898	364	1937	298	1976	208
1782	648	1821	606	1860	279	1899	422	1938	152	1977	659
1783	1782	1822	911	1861	1860	1900	399	1939	553	1978	344
1784	223	1823	1822	1862	588	1901	1900	1940	679	1979	1978
1785	84	1824	512	1863	161	1902	951	1941	647	1980	440
1786	892	1825	875	1864	1631	1903	692	1942	971	1981	566
1787	1786	1826	747	1865	745	1904	832	1943	870	1982	991
1788	447	1827	377	1866	311	1905	254	1944	1215	1983	660
1789	1788	1828	456	1867	1866	1906	952	1945	389	1984	1023
1790	179	1829	589	1868	1400	1907	1906	1946	139	1985	794
1791	198	1830	60	1869	266	1908	423	1947	176	1986	992
1792	511	1831	1830	1870	220	1909	414	1948	487	1987	1986
1793	814	1832	687	1871	1870	1910	764	1949	1948	1988	496
1794	207	1833	234	1872	351	1911	195	1950	299	1989	441

Table 9: **Values of Z(n) for n=1990(1)2223**

n	Z(n)	n	Z(n)	n	Z(n)	n	Z(n)	n	Z(n)	n	Z(n)
1990	199	2029	2028	2068	704	2107	343	2146	1072	2185	114
1991	362	2030	580	2069	2068	2108	527	2147	113	2186	1092
1992	912	2031	677	2070	459	2109	665	2148	536	2187	2186
1993	1992	2032	127	2071	436	2110	844	2149	307	2188	1640
1994	996	2033	855	2072	111	2111	2110	2150	300	2189	198
1995	189	2034	791	2073	690	2112	384	2151	477	2190	219
1996	1496	2035	110	2074	731	2113	2112	2152	1344	2191	938
1997	1996	2036	1527	2075	249	2114	755	2153	2152	2192	959
1998	296	2037	581	2076	519	2115	234	2154	359	2193	645
1999	1998	2038	1019	2077	402	2116	528	2155	430	2194	1096
2000	1375	2039	2038	2078	1039	2117	145	2156	440	2195	439
2001	551	2040	255	2079	539	2118	1059	2157	719	2196	792
2002	363	2041	156	2080	64	2119	325	2158	415	2197	2196
2003	2002	2042	1020	2081	2080	2120	159	2159	254	2198	784
2004	167	2043	908	2082	347	2121	504	2160	864	2199	732
2005	400	2044	511	2083	2082	2122	1060	2161	2160	2200	175
2006	884	2045	409	2084	520	2123	385	2162	1080	2201	496
2007	891	2046	495	2085	555	2124	648	2163	308	2202	1100
2008	752	2047	712	2086	447	2125	374	2164	1623	2203	2202
2009	245	2048	4095	2087	2086	2126	1063	2165	865	2204	551
2010	200	2049	683	2088	144	2127	708	2166	360	2205	440
2011	2010	2050	1024	2089	2088	2128	608	2167	197	2206	1103
2012	503	2051	293	2090	759	2129	2128	2168	271	2207	2206
2013	549	2052	512	2091	204	2130	284	2169	963	2208	575
2014	1007	2053	2052	2092	1568	2131	2130	2170	279	2209	2208
2015	155	2054	948	2093	91	2132	1312	2171	1001	2210	220
2016	63	2055	410	2094	348	2133	1026	2172	543	2211	66
2017	2016	2056	256	2095	419	2134	484	2173	901	2212	552
2018	1008	2057	968	2096	1440	2135	244	2174	1087	2213	2212
2019	672	2058	1028	2097	233	2136	623	2175	174	2214	1107
2020	504	2059	638	2098	1048	2137	2136	2176	255	2215	885
2021	516	2060	720	2099	2098	2138	1068	2177	622	2216	831
2022	336	2061	458	2100	224	2139	92	2178	1088	2217	738
2023	867	2062	1031	2101	572	2140	320	2179	2178	2218	1108
2024	528	2063	2062	2102	1051	2141	2140	2180	544	2219	951
2025	324	2064	128	2103	701	2142	476	2181	726	2220	480
2026	1012	2065	294	2104	1840	2143	2142	2182	1091	2221	2220
2027	2026	2066	1032	2105	420	2144	1407	2183	295	2222	1111
2028	168	2067	636	2106	324	2145	65	2184	272	2223	494

Table 10: **Values of Z(n) for n=2224(1)2457**

n	Z(n)	n	Z(n)	n	Z(n)	n	Z(n)	n	Z(n)	n	Z(n)
2224	416	2263	1022	2302	1151	2341	2340	2380	119	2419	943
2225	800	2264	848	2303	1127	2342	1171	2381	2380	2420	120
2226	371	2265	150	2304	512	2343	638	2382	396	2421	269
2227	917	2266	308	2305	460	2344	879	2383	2382	2422	692
2228	1671	2267	2266	2306	1152	2345	335	2384	447	2423	2422
2229	743	2268	567	2307	768	2346	68	2385	530	2424	303
2230	1115	2269	2268	2308	576	2347	2346	2386	1192	2425	775
2231	873	2270	680	2309	2308	2348	1760	2387	154	2426	1212
2232	495	2271	756	2310	384	2349	405	2388	1392	2427	809
2233	231	2272	639	2311	2310	2350	375	2389	2388	2428	607
2234	1116	2273	2272	2312	288	2351	2350	2390	239	2429	693
2235	149	2274	1136	2313	513	2352	735	2391	797	2430	1215
2236	559	2275	350	2314	623	2353	181	2392	207	2431	220
2237	2236	2276	568	2315	925	2354	428	2393	2392	2432	512
2238	372	2277	252	2316	192	2355	314	2394	531	2433	810
2239	2238	2278	67	2317	993	2356	247	2395	479	2434	1216
2240	384	2279	688	2318	304	2357	2356	2396	599	2435	974
2241	1079	2280	95	2319	773	2358	1179	2397	611	2436	231
2242	531	2281	2280	2320	319	2359	336	2398	1199	2437	2436
2243	2242	2282	651	2321	1055	2360	944	2399	2398	2438	1219
2244	407	2283	761	2322	215	2361	786	2400	575	2439	270
2245	449	2284	1712	2323	505	2362	1180	2401	2400	2440	304
2246	1123	2285	914	2324	664	2363	833	2402	1200	2441	2440
2247	321	2286	1016	2325	650	2364	591	2403	890	2442	296
2248	1967	2287	2286	2326	1163	2365	429	2404	600	2443	349
2249	519	2288	351	2327	715	2366	168	2405	259	2444	376
2250	999	2289	545	2328	96	2367	1052	2406	1203	2445	489
2251	2250	2290	915	2329	136	2368	1664	2407	580	2446	1223
2252	1688	2291	869	2330	699	2369	206	2408	559	2447	2446
2253	750	2292	191	2331	629	2370	315	2409	219	2448	288
2254	391	2293	2292	2332	264	2371	2370	2410	240	2449	868
2255	164	2294	1147	2333	2332	2372	592	2411	2410	2450	1175
2256	704	2295	135	2334	1167	2373	678	2412	1071	2451	171
2257	1036	2296	287	2335	934	2374	1187	2413	380	2452	1839
2258	1128	2297	2296	2336	511	2375	874	2414	1207	2453	891
2259	251	2298	383	2337	246	2376	351	2415	69	2454	408
2260	904	2299	968	2338	167	2377	2376	2416	1056	2455	490
2261	322	2300	575	2339	2338	2378	492	2417	2416	2456	1535
2262	116	2301	117	2340	584	2379	182	2418	155	2457	350

Table 11: **Values of Z(n) for n=2458(1)2691**

n	Z(n)	n	Z(n)	n	Z(n)	n	Z(n)	n	Z(n)	n	Z(n)
2458	1228	2497	681	2536	1584	2575	824	2614	1307	2653	378
2459	2458	2498	1248	2537	472	2576	160	2615	1045	2654	1327
2460	615	2499	441	2538	188	2577	858	2616	1199	2655	530
2461	321	2500	624	2539	2538	2578	1288	2617	2616	2656	2240
2462	1231	2501	122	2540	1015	2579	2578	2618	560	2657	2656
2463	821	2502	972	2541	363	2580	215	2619	485	2658	443
2464	384	2503	2502	2542	123	2581	1246	2620	655	2659	2658
2465	289	2504	2191	2543	2542	2582	1291	2621	2620	2660	399
2466	548	2505	500	2544	159	2583	287	2622	551	2661	887
2467	2466	2506	895	2545	509	2584	816	2623	731	2662	1331
2468	616	2507	436	2546	1139	2585	329	2624	1024	2663	2662
2469	822	2508	759	2547	566	2586	431	2625	125	2664	1295
2470	455	2509	1157	2548	832	2587	597	2626	403	2665	779
2471	706	2510	1004	2549	2548	2588	647	2627	851	2666	1332
2472	720	2511	1053	2550	900	2589	863	2628	72	2667	126
2473	2472	2512	1727	2551	2550	2590	259	2629	956	2668	551
2474	1236	2513	1077	2552	319	2591	2590	2630	1315	2669	628
2475	99	2514	419	2553	666	2592	1215	2631	876	2670	444
2476	1856	2515	1005	2554	1276	2593	2592	2632	1456	2671	2670
2477	2476	2516	407	2555	364	2594	1296	2633	2632	2672	1503
2478	531	2517	839	2556	71	2595	345	2634	1316	2673	242
2479	1072	2518	1259	2557	2556	2596	176	2635	340	2674	763
2480	960	2519	1144	2558	1279	2597	636	2636	1976	2675	749
2481	827	2520	224	2559	852	2598	432	2637	585	2676	1560
2482	1240	2521	2520	2560	1024	2599	1242	2638	1319	2677	2676
2483	572	2522	1260	2561	1182	2600	624	2639	377	2678	103
2484	1080	2523	840	2562	671	2601	288	2640	384	2679	798
2485	70	2524	631	2563	1165	2602	1300	2641	417	2680	335
2486	451	2525	100	2564	640	2603	684	2642	1320	2681	1148
2487	828	2526	420	2565	189	2604	216	2643	881	2682	1043
2488	2176	2527	721	2566	1283	2605	520	2644	1983	2683	2682
2489	1178	2528	1343	2567	1207	2606	1303	2645	529	2684	671
2490	995	2529	1124	2568	320	2607	395	2646	539	2685	179
2491	423	2530	275	2569	734	2608	1792	2647	2646	2686	1343
2492	623	2531	2530	2570	1284	2609	2608	2648	992	2687	2686
2493	1107	2532	632	2571	857	2610	144	2649	882	2688	1280
2494	1160	2533	595	2572	1928	2611	1119	2650	424	2689	2688
2495	499	2534	1267	2573	248	2612	1959	2651	241	2690	1075
2496	767	2535	675	2574	143	2613	402	2652	272	2691	207

Table 12: **Values of Z(n) for n=2692(1)2925**

n	Z(n)	n	Z(n)	n	Z(n)	n	Z(n)	n	Z(n)	n	Z(n)
2692	672	2731	2730	2770	1384	2809	2808	2848	2047	2887	2886
2693	2692	2732	2048	2771	815	2810	280	2849	406	2888	2527
2694	1347	2733	911	2772	440	2811	936	2850	75	2889	107
2695	440	2734	1367	2773	235	2812	703	2851	2850	2890	1155
2696	336	2735	1094	2774	1387	2813	290	2852	712	2891	294
2697	434	2736	512	2775	74	2814	335	2853	1268	2892	240
2698	284	2737	322	2776	1040	2815	1125	2854	1427	2893	263
2699	2698	2738	1368	2777	2776	2816	1023	2855	570	2894	1447
2700	999	2739	165	2778	1388	2817	1251	2856	272	2895	579
2701	73	2740	959	2779	1190	2818	1408	2857	2856	2896	543
2702	1351	2741	2740	2780	695	2819	2818	2858	1428	2897	2896
2703	476	2742	456	2781	1133	2820	375	2859	953	2898	252
2704	1183	2743	844	2782	428	2821	650	2860	935	2899	1338
2705	540	2744	2400	2783	483	2822	663	2861	2860	2900	1624
2706	164	2745	305	2784	1536	2823	941	2862	1431	2901	966
2707	2706	2746	1372	2785	1114	2824	352	2863	818	2902	1451
2708	2031	2747	737	2786	1392	2825	225	2864	895	2903	2902
2709	602	2748	687	2787	929	2826	1412	2865	764	2904	1088
2710	1084	2749	2748	2788	696	2827	770	2866	1432	2905	580
2711	2710	2750	1000	2789	2788	2828	504	2867	610	2906	1452
2712	1695	2751	392	2790	279	2829	368	2868	239	2907	152
2713	2712	2752	128	2791	2790	2830	1415	2869	151	2908	727
2714	943	2753	2752	2792	1744	2831	893	2870	615	2909	2908
2715	180	2754	323	2793	342	2832	767	2871	638	2910	1164
2716	679	2755	550	2794	1143	2833	2832	2872	2512	2911	1065
2717	208	2756	688	2795	129	2834	871	2873	169	2912	832
2718	603	2757	918	2796	1631	2835	405	2874	479	2913	971
2719	2718	2758	196	2797	2796	2836	2127	2875	874	2914	1363
2720	255	2759	712	2798	1399	2837	2836	2876	719	2915	264
2721	906	2760	575	2799	621	2838	516	2877	273	2916	728
2722	1360	2761	1254	2800	224	2839	1002	2878	1439	2917	2916
2723	777	2762	1380	2801	2800	2840	639	2879	2878	2918	1459
2724	680	2763	306	2802	467	2841	947	2880	639	2919	833
2725	1199	2764	2072	2803	2802	2842	783	2881	602	2920	1679
2726	376	2765	315	2804	2103	2843	2842	2882	131	2921	253
2727	404	2766	1383	2805	374	2844	711	2883	960	2922	1460
2728	495	2767	2766	2806	183	2845	569	2884	720	2923	1184
2729	2728	2768	864	2807	1203	2846	1423	2885	1154	2924	816
2730	104	2769	780	2808	351	2847	584	2886	480	2925	324

Table 13: **Values of Z(n) for n=2926(1)3159**

n	Z(n)	n	Z(n)	n	Z(n)	n	Z(n)	n	Z(n)	n	Z(n)
2926	76	2965	1185	3004	751	3043	357	3082	736	3121	3120
2927	2926	2966	1483	3005	600	3044	760	3083	3082	3122	223
2928	671	2967	344	3006	1503	3045	174	3084	1799	3123	693
2929	202	2968	847	3007	775	3046	1523	3085	1234	3124	1704
2930	879	2969	2968	3008	2303	3047	1385	3086	1543	3125	3124
2931	977	2970	539	3009	884	3048	1904	3087	342	3126	1563
2932	2199	2971	2970	3010	300	3049	3048	3088	192	3127	530
2933	419	2972	743	3011	3010	3050	975	3089	3088	3128	1104
2934	1304	2973	990	3012	752	3051	1242	3090	515	3129	447
2935	1174	2974	1487	3013	1310	3052	1743	3091	561	3130	939
2936	367	2975	475	3014	1232	3053	1419	3092	2319	3131	403
2937	890	2976	960	3015	134	3054	1527	3093	1031	3132	783
2938	1468	2977	1144	3016	1247	3055	234	3094	272	3133	481
2939	2938	2978	1488	3017	861	3056	191	3095	619	3134	1567
2940	440	2979	1323	3018	503	3057	1019	3096	1376	3135	209
2941	1037	2980	744	3019	3018	3058	1111	3097	1140	3136	2303
2942	1471	2981	813	3020	1359	3059	322	3098	1548	3137	3136
2943	108	2982	567	3021	741	3060	135	3099	1032	3138	1568
2944	2047	2983	627	3022	1511	3061	3060	3100	775	3139	730
2945	589	2984	1119	3023	3022	3062	1531	3101	1329	3140	784
2946	491	2985	795	3024	1728	3063	1020	3102	516	3141	1395
2947	420	2986	1492	3025	725	3064	383	3103	1391	3142	1571
2948	736	2987	927	3026	356	3065	1225	3104	3007	3143	448
2949	983	2988	1079	3027	1008	3066	875	3105	459	3144	1440
2950	1475	2989	244	3028	2271	3067	3066	3106	1552	3145	629
2951	454	2990	299	3029	233	3068	767	3107	1195	3146	363
2952	287	2991	996	3030	404	3069	341	3108	111	3147	1049
2953	2952	2992	1088	3031	433	3070	920	3109	3108	3148	2360
2954	1476	2993	656	3032	1136	3071	332	3110	1244	3149	469
2955	590	2994	1496	3033	674	3072	2048	3111	305	3150	224
2956	2216	2995	599	3034	1147	3073	1316	3112	1167	3151	137
2957	2956	2996	1176	3035	1214	3074	1536	3113	1132	3152	2560
2958	203	2997	1295	3036	528	3075	450	3114	692	3153	1050
2959	538	2998	1499	3037	3036	3076	768	3115	889	3154	664
2960	480	2999	2998	3038	588	3077	543	3116	1311	3155	630
2961	188	3000	624	3039	1013	3078	323	3117	1038	3156	263
2962	1480	3001	3000	3040	1215	3079	3078	3118	1559	3157	286
2963	2962	3002	1500	3041	3040	3080	175	3119	3118	3158	1579
2964	455	3003	77	3042	675	3081	78	3120	480	3159	728

Table 14: **Values of Z(n) for n=3160(1)3393**

n	Z(n)	n	Z(n)	n	Z(n)	n	Z(n)	n	Z(n)	n	Z(n)
3160	79	3199	1371	3238	1619	3277	1130	3316	2487	3355	549
3161	435	3200	1024	3239	1106	3278	1639	3317	1177	3356	839
3162	527	3201	582	3240	80	3279	1092	3318	315	3357	746
3163	3162	3202	1600	3241	462	3280	1024	3319	3318	3358	1679
3164	112	3203	3202	3242	1620	3281	578	3320	415	3359	3358
3165	210	3204	711	3243	1034	3282	1640	3321	81	3360	384
3166	1583	3205	640	3244	2432	3283	1273	3322	604	3361	3360
3167	3166	3206	916	3245	649	3284	2463	3323	3322	3362	1680
3168	1088	3207	1068	3246	540	3285	584	3324	831	3363	531
3169	3168	3208	400	3247	764	3286	371	3325	399	3364	840
3170	1584	3209	3208	3248	608	3287	1557	3326	1663	3365	1345
3171	755	3210	320	3249	360	3288	959	3327	1109	3366	747
3172	792	3211	1520	3250	624	3289	506	3328	2560	3367	259
3173	835	3212	583	3251	3250	3290	140	3329	3328	3368	1263
3174	528	3213	594	3252	1896	3291	1097	3330	999	3369	1122
3175	1524	3214	1607	3253	3252	3292	823	3331	3330	3370	1684
3176	1984	3215	1285	3254	1627	3293	444	3332	391	3371	3370
3177	1412	3216	1407	3255	279	3294	243	3333	605	3372	1967
3178	1588	3217	3216	3256	703	3295	659	3334	1667	3373	3372
3179	1155	3218	1608	3257	3256	3296	2368	3335	115	3374	1204
3180	159	3219	666	3258	180	3297	314	3336	416	3375	999
3181	3180	3220	160	3259	3258	3298	679	3337	141	3376	1055
3182	1332	3221	3220	3260	815	3299	3298	3338	1668	3377	307
3183	1061	3222	179	3261	1086	3300	824	3339	476	3378	563
3184	1791	3223	879	3262	699	3301	3300	3340	1335	3379	217
3185	195	3224	1456	3263	753	3302	507	3341	1027	3380	1520
3186	648	3225	300	3264	255	3303	1467	3342	1671	3381	735
3187	3186	3226	1612	3265	1305	3304	944	3343	3342	3382	1424
3188	2391	3227	461	3266	851	3305	660	3344	703	3383	1393
3189	1062	3228	807	3267	242	3306	551	3345	669	3384	1503
3190	319	3229	3228	3268	816	3307	3306	3346	1672	3385	1354
3191	3190	3230	475	3269	1400	3308	2480	3347	3346	3386	1692
3192	399	3231	359	3270	435	3309	1103	3348	216	3387	1128
3193	309	3232	1919	3271	3270	3310	1324	3349	985	3388	847
3194	1596	3233	1219	3272	2863	3311	902	3350	200	3389	3388
3195	639	3234	440	3273	1091	3312	575	3351	1116	3390	339
3196	799	3235	1294	3274	1636	3313	3312	3352	2095	3391	3390
3197	138	3236	808	3275	524	3314	1656	3353	958	3392	1536
3198	779	3237	663	3276	728	3315	390	3354	1247	3393	116

Table 15: **Values of Z(n) for n=3394(1)3627**

n	Z(n)	n	Z(n)	n	Z(n)	n	Z(n)	n	Z(n)	n	Z(n)
3394	1696	3433	3432	3472	1952	3511	3510	3550	1775	3589	776
3395	679	3434	1615	3473	1057	3512	3072	3551	1272	3590	359
3396	848	3435	915	3474	1736	3513	1170	3552	1664	3591	189
3397	473	3436	2576	3475	1250	3514	251	3553	968	3592	448
3398	1699	3437	490	3476	1264	3515	665	3554	1776	3593	3592
3399	308	3438	764	3477	854	3516	879	3555	315	3594	599
3400	799	3439	361	3478	1739	3517	3516	3556	888	3595	719
3401	1253	3440	1375	3479	1420	3518	1759	3557	3556	3596	464
3402	728	3441	185	3480	144	3519	459	3558	1779	3597	1089
3403	82	3442	1720	3481	3480	3520	384	3559	3558	3598	1028
3404	184	3443	626	3482	1740	3521	503	3560	800	3599	1769
3405	680	3444	287	3483	1376	3522	587	3561	1187	3600	224
3406	1572	3445	689	3484	871	3523	1625	3562	1507	3601	831
3407	3406	3446	1723	3485	204	3524	880	3563	1526	3602	1800
3408	639	3447	765	3486	83	3525	375	3564	648	3603	1200
3409	973	3448	431	3487	1584	3526	860	3565	620	3604	424
3410	340	3449	3448	3488	2943	3527	3526	3566	1783	3605	720
3411	378	3450	275	3489	1163	3528	783	3567	492	3606	600
3412	2559	3451	203	3490	1395	3529	3528	3568	223	3607	3606
3413	3412	3452	863	3491	3490	3530	1059	3569	1161	3608	1968
3414	1707	3453	1151	3492	872	3531	428	3570	84	3609	801
3415	1365	3454	627	3493	1497	3532	2648	3571	3570	3610	360
3416	671	3455	690	3494	1747	3533	3532	3572	1880	3611	942
3417	1071	3456	512	3495	465	3534	588	3573	396	3612	903
3418	1708	3457	3456	3496	1311	3535	504	3574	1787	3613	3612
3419	1052	3458	455	3497	806	3536	832	3575	649	3614	416
3420	360	3459	1152	3498	264	3537	917	3576	447	3615	240
3421	1243	3460	519	3499	3498	3538	1159	3577	146	3616	1920
3422	1652	3461	3460	3500	1000	3539	3538	3578	1788	3617	3616
3423	489	3462	576	3501	1556	3540	944	3579	1193	3618	1808
3424	320	3463	3462	3502	1648	3541	3540	3580	895	3619	329
3425	274	3464	432	3503	1581	3542	252	3581	3580	3620	904
3426	1712	3465	440	3504	1824	3543	1181	3582	1592	3621	425
3427	298	3466	1732	3505	700	3544	1328	3583	3582	3622	1811
3428	856	3467	3466	3506	1752	3545	709	3584	3072	3623	3622
3429	1269	3468	288	3507	167	3546	1575	3585	239	3624	1056
3430	1715	3469	3468	3508	2631	3547	3546	3586	1792	3625	1624
3431	657	3470	1040	3509	725	3548	887	3587	1054	3626	147
3432	143	3471	533	3510	324	3549	168	3588	207	3627	558

Table 16: **Values of Z(n) for n=3628(1)3861**

n	Z(n)	n	Z(n)	n	Z(n)	n	Z(n)	n	Z(n)	n	Z(n)
3628	2720	3667	1158	3706	544	3745	749	3784	1375	3823	3822
3629	190	3668	392	3707	1011	3746	1872	3785	1514	3824	3584
3630	120	3669	1223	3708	720	3747	1248	3786	1892	3825	900
3631	3630	3670	1100	3709	3708	3748	936	3787	1623	3826	1912
3632	2496	3671	3670	3710	1484	3749	1793	3788	2840	3827	1246
3633	518	3672	1376	3711	1236	3750	624	3789	1683	3828	87
3634	552	3673	3672	3712	1536	3751	1209	3790	379	3829	546
3635	1454	3674	835	3713	1738	3752	335	3791	1784	3830	1915
3636	504	3675	1175	3714	1856	3753	972	3792	1184	3831	1277
3637	3636	3676	919	3715	1485	3754	1876	3793	3792	3832	479
3638	748	3677	3676	3716	928	3755	750	3794	812	3833	3832
3639	1212	3678	612	3717	413	3756	312	3795	230	3834	567
3640	559	3679	1131	3718	1352	3757	1156	3796	584	3835	649
3641	330	3680	575	3719	3718	3758	1879	3797	3796	3836	959
3642	1820	3681	818	3720	464	3759	357	3798	1476	3837	1278
3643	3642	3682	1315	3721	3720	3760	704	3799	261	3838	303
3644	911	3683	1015	3722	1860	3761	3760	3800	399	3839	1396
3645	729	3684	920	3723	510	3762	836	3801	1085	3840	1535
3646	1823	3685	670	3724	1519	3763	212	3802	1900	3841	667
3647	1042	3686	1843	3725	149	3764	2823	3803	3802	3842	339
3648	512	3687	1229	3726	1863	3765	1004	3804	951	3843	854
3649	533	3688	2304	3727	3726	3766	1344	3805	760	3844	960
3650	875	3689	713	3728	1631	3767	3766	3806	692	3845	769
3651	1217	3690	819	3729	791	3768	1727	3807	1457	3846	1923
3652	912	3691	3690	3730	1119	3769	3768	3808	832	3847	3846
3653	1404	3692	1703	3731	286	3770	260	3809	585	3848	480
3654	783	3693	1230	3732	311	3771	837	3810	380	3849	1283
3655	85	3694	1847	3733	3732	3772	368	3811	1442	3850	175
3656	3199	3695	739	3734	1867	3773	1715	3812	952	3851	3850
3657	689	3696	384	3735	414	3774	407	3813	123	3852	855
3658	1239	3697	3696	3736	2335	3775	150	3814	1907	3853	3852
3659	3658	3698	1848	3737	1110	3776	767	3815	545	3854	328
3660	975	3699	1781	3738	356	3777	1259	3816	1007	3855	770
3661	1568	3700	999	3739	3738	3778	1888	3817	693	3856	3615
3662	1831	3701	3700	3740	560	3779	3778	3818	1908	3857	608
3663	296	3702	1851	3741	86	3780	944	3819	1139	3858	1928
3664	2976	3703	1057	3742	1871	3781	398	3820	1719	3859	680
3665	1465	3704	463	3743	1576	3782	1891	3821	3820	3860	1544
3666	611	3705	285	3744	1664	3783	194	3822	195	3861	351

Table 17: **Values of Z(n) for n=3862(1)4095**

n	Z(n)	n	Z(n)	n	Z(n)	n	Z(n)	n	Z(n)	n	Z(n)
3862	1931	3901	1410	3940	984	3979	345	4018	1763	4057	4056
3863	3862	3902	1951	3941	1126	3980	199	4019	4018	4058	2028
3864	735	3903	1301	3942	1971	3981	1326	4020	200	4059	450
3865	1545	3904	1280	3943	3942	3982	1628	4021	4020	4060	840
3866	1932	3905	780	3944	2175	3983	1707	4022	2011	4061	1178
3867	1289	3906	216	3945	525	3984	2240	4023	297	4062	2031
3868	967	3907	3906	3946	1972	3985	1594	4024	3520	4063	238
3869	583	3908	976	3947	3946	3986	1992	4025	574	4064	127
3870	215	3909	1302	3948	1127	3987	1772	4026	671	4065	270
3871	1421	3910	459	3949	1077	3988	2991	4027	4026	4066	855
3872	1088	3911	3910	3950	1500	3989	3988	4028	1007	4067	1078
3873	1290	3912	815	3951	1755	3990	399	4029	237	4068	791
3874	1936	3913	559	3952	1728	3991	1534	4030	155	4069	312
3875	124	3914	1235	3953	1474	3992	2495	4031	695	4070	924
3876	152	3915	405	3954	659	3993	1331	4032	1791	4071	413
3877	3876	3916	88	3955	790	3994	1996	4033	1961	4072	2544
3878	1939	3917	3916	3956	344	3995	799	4034	2016	4073	4072
3879	431	3918	1959	3957	1319	3996	296	4035	269	4074	776
3880	1455	3919	3918	3958	1979	3997	1141	4036	1008	4075	325
3881	3880	3920	735	3959	962	3998	1999	4037	1100	4076	3056
3882	647	3921	1307	3960	495	3999	558	4038	672	4077	755
3883	352	3922	1960	3961	1631	4000	2624	4039	1154	4078	2039
3884	2912	3923	3922	3962	1980	4001	4000	4040	1615	4079	4078
3885	629	3924	872	3963	1320	4002	551	4041	449	4080	255
3886	1072	3925	1099	3964	991	4003	4002	4042	516	4081	847
3887	506	3926	1208	3965	610	4004	1000	4043	311	4082	156
3888	1215	3927	153	3966	660	4005	89	4044	336	4083	1361
3889	3888	3928	1472	3967	3966	4006	2003	4045	809	4084	3063
3890	1555	3929	3928	3968	1023	4007	4006	4046	867	4085	645
3891	1296	3930	524	3969	1862	4008	1503	4047	284	4086	908
3892	1112	3931	3930	3970	1984	4009	1899	4048	736	4087	670
3893	458	3932	983	3971	1804	4010	400	4049	4048	4088	511
3894	176	3933	873	3972	992	4011	573	4050	324	4089	986
3895	779	3934	280	3973	1507	4012	1887	4051	4050	4090	1635
3896	3408	3935	1574	3974	1987	4013	4012	4052	3039	4091	4090
3897	432	3936	1599	3975	900	4014	891	4053	965	4092	495
3898	1948	3937	1270	3976	496	4015	219	4054	2027	4093	4092
3899	1113	3938	715	3977	1066	4016	3263	4055	810	4094	712
3900	624	3939	909	3978	611	4017	1235	4056	1520	4095	90

Table 18: **Values of Z(n) for n=4096(1)4329**

n	Z(n)	n	Z(n)	n	Z(n)	n	Z(n)	n	Z(n)	n	Z(n)
4096	8191	4135	1654	4174	2087	4213	1914	4252	1063	4291	1225
4097	1445	4136	704	4175	500	4214	343	4253	4252	4292	1072
4098	683	4137	1182	4176	927	4215	1124	4254	708	4293	1377
4099	4098	4138	2068	4177	4176	4216	527	4255	184	4294	2147
4100	1024	4139	4138	4178	2088	4217	4216	4256	1728	4295	859
4101	1367	4140	575	4179	398	4218	740	4257	945	4296	1968
4102	2051	4141	1312	4180	759	4219	4218	4258	2128	4297	4296
4103	373	4142	436	4181	2034	4220	1055	4259	4258	4298	307
4104	512	4143	1380	4182	204	4221	468	4260	639	4299	1433
4105	820	4144	1183	4183	1691	4222	2111	4261	4260	4300	1375
4106	2052	4145	829	4184	1568	4223	205	4262	2131	4301	781
4107	1368	4146	2072	4185	620	4224	1023	4263	783	4302	2151
4108	1975	4147	637	4186	91	4225	675	4264	1312	4303	662
4109	587	4148	1768	4187	158	4226	2112	4265	1705	4304	1344
4110	684	4149	1844	4188	1047	4227	1409	4266	2132	4305	245
4111	4110	4150	2075	4189	354	4228	1056	4267	1003	4306	2152
4112	256	4151	1778	4190	419	4229	4228	4268	1551	4307	1533
4113	1827	4152	864	4191	1143	4230	1080	4269	1422	4308	359
4114	968	4153	4152	4192	2751	4231	4230	4270	244	4309	278
4115	1645	4154	2076	4193	1197	4232	528	4271	4270	4310	1724
4116	2400	4155	554	4194	1863	4233	663	4272	800	4311	1916
4117	1610	4156	1039	4195	839	4234	1971	4273	4272	4312	1616
4118	1420	4157	4156	4196	1048	4235	1210	4274	2136	4313	227
4119	1373	4158	539	4197	1398	4236	2471	4275	1025	4314	719
4120	720	4159	4158	4198	2099	4237	892	4276	3207	4315	1725
4121	1585	4160	1664	4199	493	4238	2119	4277	987	4316	415
4122	2060	4161	437	4200	224	4239	1727	4278	92	4317	1439
4123	588	4162	2080	4201	4200	4240	159	4279	1166	4318	1904
4124	1031	4163	1448	4202	572	4241	4240	4280	320	4319	616
4125	374	4164	1040	4203	467	4242	504	4281	1427	4320	1215
4126	2063	4165	1224	4204	3152	4243	4242	4282	2140	4321	1043
4127	4126	4166	2083	4205	840	4244	3183	4283	4282	4322	2160
4128	128	4167	926	4206	2103	4245	849	4284	1071	4323	131
4129	4128	4168	3647	4207	601	4246	2123	4285	1714	4324	1080
4130	944	4169	758	4208	2367	4247	1643	4286	2143	4325	2075
4131	1700	4170	555	4209	183	4248	944	4287	1428	4326	308
4132	1032	4171	387	4210	420	4249	1820	4288	1407	4327	4326
4133	4132	4172	447	4211	4210	4250	2124	4289	4288	4328	2704
4134	636	4173	428	4212	728	4251	545	4290	780	4329	702

§3. Some remarks

The following conjecture has been made by Wengpeng Zhang and Ling Li [7] :

Conjecture 2. For any integer $n \geq 1$, the equation S_c(n) =Z_*(n) + n has the only solutions

$$n = p^{2m-1},$$

where $p \geq 5$ is a prime and $m \geq 1$ is an integer such that p^{2m-1} and $p^{2m-1} + 2$ are twin primes.

Though the numerical values given in the tables supports Conjecture 1 of Wengpeng Zhang and Ling Li [7], we found several counter-examples to the "only if" part of Conjecture 2. For examples,

$S_c(35)$ =36 =$Z_*(35) + 35$, $S_c(65)$ =66 =$Z_*(65) + 65$, $S_c(77)$ =78 =$Z_*(77) + 77$.

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Table 19: **Values of Z(n) for n=4330(1)4563**

n	Z(n)	n	Z(n)	n	Z(n)	n	Z(n)	n	Z(n)	n	Z(n)
4330	1299	4369	2056	4408	608	4447	4446	4486	2243	4525	724
4331	426	4370	759	4409	4408	4448	4031	4487	1281	4526	1240
4332	360	4371	93	4410	440	4449	1482	4488	527	4527	503
4333	1238	4372	3279	4411	802	4450	800	4489	4488	4528	3679
4334	2167	4373	4372	4412	1103	4451	4450	4490	1795	4529	1294
4335	1155	4374	2187	4413	1470	4452	1007	4491	998	4530	755
4336	4064	4375	1875	4414	2207	4453	365	4492	3368	4531	1379
4337	4336	4376	2735	4415	1765	4454	2227	4493	4492	4532	824
4338	963	4377	1458	4416	896	4455	890	4494	1176	4533	1511
4339	4338	4378	2188	4417	630	4456	2784	4495	434	4534	2267
4340	279	4379	754	4418	2208	4457	4456	4496	2528	4535	1814
4341	1446	4380	584	4419	981	4458	743	4497	1499	4536	1295
4342	2171	4381	337	4420	935	4459	1715	4498	519	4537	2093
4343	2020	4382	1252	4421	4420	4460	1560	4499	2045	4538	2268
4344	543	4383	486	4422	803	4461	1487	4500	999	4539	356
4345	395	4384	959	4423	4422	4462	2231	4501	643	4540	680
4346	1271	4385	1754	4424	1343	4463	4462	4502	2251	4541	1672
4347	161	4386	731	4425	825	4464	1952	4503	474	4542	756
4348	1087	4387	1926	4426	2212	4465	94	4504	2815	4543	825
4349	4348	4388	1096	4427	931	4466	231	4505	424	4544	639
4350	899	4389	132	4428	2295	4467	1488	4506	2252	4545	404
4351	228	4390	439	4429	515	4468	3351	4507	4506	4546	2272
4352	4096	4391	4390	4430	2215	4469	327	4508	391	4547	4546
4353	1451	4392	1647	4431	210	4470	744	4509	1836	4548	1136
4354	2176	4393	1909	4432	831	4471	526	4510	164	4549	4548
4355	870	4394	2196	4433	1209	4472	559	4511	1040	4550	1000
4356	1088	4395	585	4434	2216	4473	567	4512	704	4551	369
4357	4356	4396	784	4435	1774	4474	2236	4513	4512	4552	3983
4358	2179	4397	4396	4436	3327	4475	1074	4514	1036	4553	1884
4359	1452	4398	732	4437	782	4476	1119	4515	300	4554	252
4360	544	4399	1908	4438	951	4477	1331	4516	1128	4555	910
4361	979	4400	1375	4439	965	4478	2239	4517	4516	4556	1071
4362	2180	4401	162	4440	480	4479	1493	4518	251	4557	588
4363	4362	4402	496	4441	4440	4480	1280	4519	4518	4558	688
4364	3272	4403	629	4442	2220	4481	4480	4520	1695	4559	1551
4365	485	4404	2568	4443	1481	4482	1079	4521	1232	4560	95
4366	295	4405	880	4444	1111	4483	4482	4522	475	4561	4560
4367	396	4406	2203	4445	889	4484	1120	4523	4522	4562	2280
4368	896	4407	338	4446	1196	4485	299	4524	1247	4563	675

Table 20: **Values of Z(n) for n=4564(1)4797**

n	Z(n)	n	Z(n)	n	Z(n)	n	Z(n)	n	Z(n)	n	Z(n)
4564	1792	4603	4602	4642	1055	4681	1208	4720	1120	4759	4758
4565	165	4604	1151	4643	4642	4682	2340	4721	4720	4760	560
4566	2283	4605	614	4644	215	4683	1337	4722	2360	4761	2115
4567	4566	4606	1127	4645	929	4684	3512	4723	4722	4762	2380
4568	1712	4607	271	4646	2323	4685	1874	4724	3543	4763	1298
4569	1523	4608	4095	4647	1548	4686	780	4725	350	4764	1191
4570	2284	4609	418	4648	1743	4687	1634	4726	2363	4765	1905
4571	1959	4610	460	4649	4648	4688	3808	4727	1304	4766	2383
4572	1016	4611	318	4650	899	4689	521	4728	591	4767	1134
4573	1614	4612	1152	4651	4650	4690	335	4729	4728	4768	447
4574	2287	4613	658	4652	3488	4691	4690	4730	1375	4769	1254
4575	549	4614	768	4653	846	4692	1104	4731	912	4770	900
4576	2431	4615	780	4654	715	4693	1443	4732	168	4771	1468
4577	597	4616	576	4655	930	4694	2347	4733	4732	4772	1192
4578	980	4617	1215	4656	96	4695	939	4734	1052	4773	258
4579	722	4618	2308	4657	4656	4696	1760	4735	1894	4774	868
4580	1144	4619	744	4658	136	4697	671	4736	3071	4775	2100
4581	1017	4620	384	4659	1553	4698	1943	4737	1578	4776	1392
4582	2291	4621	4620	4660	1864	4699	888	4738	2368	4777	561
4583	4582	4622	2311	4661	236	4700	375	4739	2030	4778	2388
4584	191	4623	804	4662	1035	4701	1566	4740	1184	4779	648
4585	524	4624	288	4663	4662	4702	2351	4741	2155	4780	239
4586	2292	4625	999	4664	847	4703	4702	4742	2371	4781	1365
4587	417	4626	1799	4665	1244	4704	2303	4743	1053	4782	2391
4588	2479	4627	1322	4666	2332	4705	940	4744	592	4783	4782
4589	2118	4628	623	4667	1794	4706	2171	4745	364	4784	896
4590	135	4629	1542	4668	1167	4707	522	4746	791	4785	725
4591	4590	4630	2315	4669	1218	4708	1176	4747	2020	4786	2392
4592	287	4631	1683	4670	1400	4709	1938	4748	3560	4787	4786
4593	1530	4632	192	4671	864	4710	1884	4749	1583	4788	1007
4594	2296	4633	451	4672	511	4711	672	4750	1500	4789	4788
4595	919	4634	1323	4673	4672	4712	1519	4751	4750	4790	479
4596	383	4635	720	4674	779	4713	1571	4752	351	4791	1596
4597	4596	4636	304	4675	374	4714	2356	4753	97	4792	4192
4598	968	4637	4636	4676	167	4715	574	4754	2376	4793	4792
4599	657	4638	2319	4677	1559	4716	1440	4755	950	4794	611
4600	575	4639	4638	4678	2339	4717	2225	4756	696	4795	685
4601	214	4640	319	4679	4678	4718	336	4757	1206	4796	1199
4602	767	4641	272	4680	1520	4719	363	4758	792	4797	819

Table 21: **Values of Z(n) for n=4798(1)5000**

n	Z(n)	n	Z(n)	n	Z(n)	n	Z(n)	n	Z(n)	n	Z(n)
4798	2399	4832	3775	4866	2432	4900	1175	4934	2467	4968	2943
4799	4798	4833	1431	4867	2355	4901	1014	4935	140	4969	4968
4800	2175	4834	2416	4868	1216	4902	171	4936	4319	4970	1064
4801	4800	4835	1934	4869	540	4903	4902	4937	4936	4971	1656
4802	2400	4836	1208	4870	1460	4904	1839	4938	2468	4972	791
4803	1601	4837	2072	4871	4870	4905	980	4939	2244	4973	4972
4804	1200	4838	943	4872	608	4906	891	4940	455	4974	828
4805	960	4839	1613	4873	1771	4907	700	4941	243	4975	199
4806	1512	4840	1935	4874	2436	4908	408	4942	1764	4976	2176
4807	759	4841	2162	4875	624	4909	4908	4943	4942	4977	315
4808	4207	4842	2151	4876	2967	4910	1964	4944	927	4978	2488
4809	686	4843	667	4877	4876	4911	1637	4945	344	4979	766
4810	259	4844	1903	4878	2168	4912	1535	4946	2472	4980	1079
4811	849	4845	170	4879	1189	4913	4912	4947	969	4981	1172
4812	2807	4846	2423	4880	1280	4914	728	4948	3711	4982	423
4813	4812	4847	1702	4881	1626	4915	1965	4949	1616	4983	1056
4814	580	4848	1919	4882	2440	4916	3687	4950	99	4984	623
4815	855	4849	1118	4883	513	4917	297	4951	4950	4985	1994
4816	1504	4850	775	4884	296	4918	2459	4952	1856	4986	1107
4817	4816	4851	98	4885	1954	4919	4818	4953	507	4987	4986
4818	219	4852	3639	4886	2443	4920	1599	4954	2476	4988	1160
4819	1342	4853	1265	4887	1809	4921	665	4955	990	4989	1662
4820	240	4854	2427	4888	1456	4922	2139	4956	944	4990	499
4821	1607	4855	970	4889	4888	4923	2187	4957	4956	4991	713
4822	2411	4856	607	4890	815	4924	1231	4958	1072	4992	767
4823	636	4857	1619	4891	803	4925	1575	4959	1044	4993	4992
4824	1071	4858	1735	4892	1223	4926	2463	4960	960	4994	1815
4825	1350	4859	903	4893	699	4927	2274	4961	2419	4995	999
4826	380	4860	1215	4894	2447	4928	384	4962	827	4996	1248
4827	1608	4861	4860	4895	890	4929	371	4963	2127	4997	1577
4828	1207	4862	220	4896	1088	4930	1275	4964	1240	4998	1224
4829	439	4863	1620	4897	2241	4931	4930	4965	330	4999	4998
4830	644	4864	512	4898	868	4932	1232	4966	572	5000	624
4831	4830	4865	139	4899	851	4933	4932	4967	4966		

m-graphoidal path covers of a graph

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Abstract The concept of graphoidal cover was introduced by B. D. Acharya and E. Sampathkumar and 2-graphoidal path cover of a graph was introduced by K. Nagarajan, A. Nagarajan and S. Somasundaramn. In this paper, we define a m-graphoidal path cover of a graph. A m-graphoidal path cover of a graph G is a partition of edge set of G into paths in which every vertex is an internal vertex of at most m paths and the minimum cardinality of it is a m-graphoidal path covering number $\eta_m(G)$. In this paper we initiate a study of the parameter $\eta_m(G)$ and we find $\eta_m(G)$ for some standard graphs. Also we find for which m, the parameters $\eta_m(G)$ and the path partition number $\pi(G)$ are the same for wheels, complete graphs and stars.

Keywords Graphoidal cover, 2-graphoidal path cover, m-graphoidal path cover, m-graphoidal path covering number, Path partition number.

§1. Introduction

By a graph, we mean a finite, undirected, non-trivial, connected graph without loops and multiple edges. The order and size of a graph are denoted by p and q respectively. For terms not defined here we refer to Harary [3].

Let $P = (v_1, v_2, \cdots, v_n)$ be a path in a graph $G = (V, E)$. The vertices $v_2, v_3, \cdots, v_{n-1}$ are called internal vertices of P and v_1 and v_n are called external vertices of P.

A decomposition of a graph G is a collection of edge-disjoint subgraphs $H_1, H_2, \cdots, H_r$ of G such that every edge of G belongs to exactly one H_i.

Definition 1.1. If each H_i is a path, then ψ is called a path partition or path cover of G. The minimum cardinality of a path partition of G is called the path partition number of G and is denoted by $\pi(G)$ and any path partition ψ of G for which $|\psi| = \pi(G)$ is called a minimum path partition or π-cover of G.

The parameter π was studied by Harary and Schwenk [4], Stanton et.al., [8] and Arumugam and Suresh Suseela [2]. Path partition numbers of some standard graphs are given in the following theorem.

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Theorem 1.2. [8]

(a) $\pi(K_p) = \left\lceil \frac{p}{2} \right\rceil$ (b) $\pi(W_n) = \left\lfloor \frac{n}{2} \right\rfloor$ and (c) $\pi(K_{1,n}) = \left\lceil \frac{n}{2} \right\rceil$.

Definition 1.3. [7] A graphoidal cover ψ of a graph G is a partition of $E(G)$ into non-trivial (not necessarily open) paths in G such that every vertex of G is an internal vertex of at most one path in ψ.

The minimum cardinality of a graphoidal cover of G is called the graphoidal covering number of G. C. Pakkiam and S. Arumugam [6] have determined the graphoidal covering number of several families of graphs. B. D. Acharya [1] studied the results on the graphoidal covering number of a graph. Further S. Arumugam and J. Suresh Suseela [2] introduced the concept of acyclic graphoidal cover.

Definition 1.4. [2] An acyclic graphoidal cover of G is a graphoidal cover ψ of G such that every element of ψ is a path in G. The minimum cardinality of an acyclic graphoidal cover of G is called the acyclic graphoidal covering number of G and is denoted by $\eta_a(G)$.

Further K. Nagarajan , A. Nagarajan and S. Somasundaram [5] introduced the concept of 2-graphoidal path cover.

Definition 1.5. [5] A 2-graphoidal path cover ψ is a collection of non-trivial paths in G such that

(i) Every edge is in exactly one path in ψ.

(ii) Every vertex is an internal vertex of at most two paths in ψ.

Let $\mathfrak{g}_2$ denote the collection of all 2-graphoidal path covers of G. Since $E(G)$ is a 2-graphoidal path cover, we have $\mathfrak{g}_2 \neq \phi$. The minimum cardinality of a 2-graphoidal path cover of G is called the 2-graphoidal path covering number of G and is denoted by $\eta_2(G)$.

That is $\eta_2(G)= \min\{|\psi| : \psi \in g_2(G)\}$.

§2. Main results

The concept of graphoidal cover was introduced by B. D. Acharya and E. Sampathkumar [7] and K. Nagarajan , A. Nagarajan and S. Somasundaram [5] introduced the concept of 2-graphoidal path cover. It motivates us to define the generalized graphoidal path cover called an m-graphoidal path cover of G, which is defined as follows.

Definition 2.1. An m-graphoidal path cover ψ is a collection of non-trivial paths in G such that

(i) Every edge is in exactly one path in ψ.

(ii) Every vertex is an internal vertex of at most m paths in ψ.

Let $\mathfrak{g}_m$ denote the collection of all m-graphoidal path covers of G. Since $E(G)$ is a m-graphoidal path cover, we have $\mathfrak{g}_m \neq \phi$. The minimum cardinality of an m-graphoidal path cover of G is called the m-graphoidal path covering number of G and is denoted by $\eta_m(G)$.

Thus $\eta_m(G)= \min\{|\psi| : \psi \in g_m(G)\}$.

Remark 2.2. (1) We also call acyclic graphoidal cover as 1- graphoidal path cover. Hereafter we denote the acyclic graphoidal covering number of G by $\eta_1(G)$ or simply η_1 when there is no possibility of confusion.
(2) If $m = 2$, then we get a 2-graphoidal path cover [5]. The 2-graphoidal path covering number η_2 was studied in [5].
(3) If $l \leq m$, then every l-graphoidal path cover is an m-graphoidal path cover and consequently, $\eta_m \leq \eta_l$.

From the definition, we derive the following observations.

Observation 2.3. For any graph G, $1 \leq \eta_m(G) \leq q$. Also $\eta_m(G) = 1$ if and only if G is a path and $\eta_m(G) = q$ if and only if $G \cong K_2$.

Observation 2.4. For any graph G, $\eta_m(G) \geq \Delta - m$.

Observation 2.5. Since there exists a vertex in G which is internal in at most $\lceil \frac{\Delta}{2} \rceil$ paths, we have $m \leq \lceil \frac{\Delta}{2} \rceil$.

Next we give some notations, which will be used to simplify the proofs of the theorems.

Notations 2.6. Let Ψ be an m-graphoidal path cover of G.
i_Ψ(P)-number of internal vertices of the path P in Ψ.
$t_i(\Psi)$-number of internal vertices which appear exactly i times in the paths of Ψ of G for $i = 1, 2, \cdots, m$.
t_Ψ-number of vertices which are not internal in any path of Ψ.

If an m-graphoidal path cover Ψ of G is minimum, then clearly $t_i(\Psi)$ should be maximum and t_Ψ should be minimum. So we define the following:
$t_i = max\ t_i(\Psi)$ where the maximum is taken from all m-graphoidal path covers Ψ of G for $i = 1, 2, \cdots, m$ and $t = min\ t_\Psi$ where the minimum is taken from all m-graphoidal path covers Ψ of G.

The following theorem gives the lower bound for the parameter η_m.

Theorem 2.7. Let G be a (p, q) graph. Then $q - mp \leq \eta_m(G)$.

Proof. Suppose $\{P_1, P_2, \cdots, P_k\}$ is a minimum m-graphoidal path cover of G. Then

$$\begin{aligned} |E(G)| &= \sum_{j=1}^{k} |E(P_j)| \\ &= \sum_{j=1}^{k} (i_\Psi(P_j) + 1) \\ &= k + \sum_{j=1}^{k} i_\Psi(P_j). \end{aligned}$$

Since every vertex of G is an internal vertex of at most m paths, we have

$$\begin{aligned}
\sum_{j=1}^{k} i_{\Psi}(P_j) &\leq m\,|V(G)| \\
Thus\,|E(G)| &\leq k + m\,|V(G)| \\
&= \eta_m(G) + m\,|V(G)| \\
|E(G)| - m\,|V(G)| &\leq \eta_m(G) \\
q - mp &\leq \eta_m(G).
\end{aligned}$$

Next we present a general result which is useful in determining the value of η_m for some standard graphs.

Theorem 2.8. For any graph G, $\eta_m(G) = q - p - \sum_{k=2}^{m}(k-1)t_k + t$

Proof. For any $\psi \in \mathfrak{g}_{\mathbf{m}}$, we have

$$\begin{aligned}
q &= \sum |E(P)| \\
&= \sum (i_\psi(P) + 1) \\
&= \sum i_\psi(P) + |\psi| \\
&= \sum_{k=1}^{m} k t_k(\psi) + |\psi| \\
&= \sum_{k=2}^{m}(k-1)t_k(\psi) + \sum_{k=1}^{m} t_k(\psi) + |\psi| \\
&= \sum_{k=2}^{m}(k-1)t_k(\psi) + (p - t_\psi) + |\psi| \\
|\psi| &= q - p - \sum_{k=2}^{m}(k-1)t_k(\psi) + t_\psi.
\end{aligned}$$

Thus, $\eta_m(G) = q - p - \sum_{k=2}^{m}(k-1)t_k + t$.

Corollary 2.9. [2] For any (p, q) graph G, $\eta_1(G) = q - p + t$.

Corollary 2.10. [5] For any (p, q) graph G, $\eta_2(G) = q - p - t_2 + t$.

Corollary 2.11. For any (p, q) graph G, $\eta_m(G) \geq q - p - \sum_{k=2}^{m}(k-1)t_k$.

Corollary 2.12. For any (p, q) graph G, the following are equivalent.

(a) $\eta_m(G) = q - p - \sum_{k=2}^{m}(k-1)t_k$ (b) There exists an m-graphoidal path cover in which every vertex is an internal vertex of a path in ψ .

Corollary 2.13. There exists a m-graphoidal path cover ψ of G in which every vertex is an internal vertex of exactly m paths in ψ of G if and only if $\eta_m(G) = q - mp$.

Proof. Since every vertex is an internal vertex of exactly m paths in ψ of G, $t_2 = t_3 = \cdots = t_{m-1} = 0$ and $t_m = p$. Then the result follows from Corollary 2.12.

Corollary 2.14. For a graph G with $\Delta(G) \leq 3$, $\eta_m(G) = \eta_l(G)$ for $l = 1, 2, \cdots, m-1$.

Proof. Since $\Delta \leq 3$, $t_k = 0$ for all $k = 2, \cdots, m-1$. Then the result follows from Theorem 2.8 and Corollary 2.12.

Corollary 2.15. For a graph G with $\Delta(G) \leq 2m-1$, $\eta_m(G) = \eta_{m-1}(G)$.

Proof. Since $\Delta \leq 2m-1$, $t_m = 0$. Then from Theorem 2.8, it follows that $\eta_m(G) = q - p - \sum_{k=2}^{m-1}(k-1)t_k + t = \eta_{m-1}(G)$.

Corollary 2.16. Let G be any (p,q) graph such that $\eta_m(G) = q - mp$. Then $\delta(G) \geq 2m$.

Proof. By the Corollary 2.13, there exists an m-graphoidal path cover Ψ of G in which every vertex is an internal vertex of exactly m paths in ψ and so $\delta \geq 2m$.

Corollary 2.17. If G is a (p,q) graph with $\eta_m(G) = q - mp$, then $\Delta(G) \geq 2m+1$.

Proof. By Corollary 2.16, $d(v) \geq 2m$ for all $v \in V(G)$. Suppose there is no vertex of degree exceeding $2m$, G must be $2m$-regular so, we have $q = mp$ and $\eta_m(G) = mp - mp = 0$, which is a contradiction. Hence $\Delta(G) \geq 2m+1$.

The following theorem gives that the necessary condition for the equality of the parameters η_m and π.

Theorem 2.18. If G is a (p,q) graph with $\Delta \leq 2m+1$, then $\eta_m(G) = \pi(G)$.

Proof. Since every m-graphoidal path cover is a path cover we have $\pi(G) \leq \eta_m(G)$. If $\Delta \leq 2m+1$, then every path cover is an m-graphoidal path cover of G and hence $\eta_m(G) = \pi(G)$.

We give the sufficient condition as an open problem.

Problem 2.19. If G is a (p,q) graph with $\eta_m(G) = \pi(G)$, then $\Delta \leq 2m+1$.

In the following theorems we determine the value of η_m of several classes of graphs such as wheel and complete graphs and also investigate forwhich m the parameters η_m and π are equal.

Theorem 2.20. For a wheel $W_n = K_1 + C_{n-1} (n \geq 4)$,

$$\eta_m(W_n) = \begin{cases} n-(m+1) & \text{if} \quad n \geq 2m+1 \\ n-m & \text{if} \quad n \leq 2m. \end{cases}$$

Proof. Let $V(W_n) = \{v_1, v_2, \ldots, v_{n-1}, v_n\}$ and let $E(W_n) = \{v_iv_{i+1} : 1 \leq i \leq n-1\} \bigcup \{v_1v_{n-1}\} \bigcup \{v_nv_i : 1 \leq i \leq n-1\}$. Since $d(v_i) = 3$ $(i = 1, 2, \cdots, n-1)$ and $d(v_n) = n-1$, the vertices $v_1, v_2, \cdots, v_{n-1}$ are internal in at most one path and the vertex v_n is internal in at most m paths and so $t_k = 0$ for $k = 2, 3, \cdots, m-1$ and $t_m \leq 1$. From Corollary 2.11 it follows that

$$\begin{aligned} \eta_m(W_n) &\geq q - p - (m-1)t_m \\ &\geq 2n - 2 - n - (m-1) \\ &= n - (m+1) \end{aligned}$$

Suppose $n > 2m+1$. Now let
$P = (v_1, v_2, \cdots, v_{n-1}, v_n)$
$P_1 = (v_{n-1}, v_1, v_n, v_2)$

$P_i = (v_{2i-1}, v_n, v_{2i})$, $2 \le i \le m$

Then the paths $P, P_1, P_2, \cdots, P_m$ together with the remaining edges form an m-graphoidal path cover ψ of W_n and so

$$\begin{aligned} |\psi| &= m + 1 + |E(W_n) - \{E(P) \cup E(P_1) \cup E(P_2) \cup E(P_3) \ldots E(P_m)\}| \\ &= m + 1 + (2n-2) - (n - 1 + 3 + \underbrace{2 + 2 + \cdots 2}_{(m-1)-\text{times}}) \\ &= n - (m+1). \\ \text{and} \qquad \eta_m(W_n) &\le n - (m+1) \end{aligned}$$

Hence $\eta_m(W_n) = n - (m+1)$ for $n > 2m+1$.

Now suppose $n = 2m+1$. Let
$P_i = (v_{i+1}, v_i, v_{2m+1}, v_{m+i}, v_{m+1+i}), 1 \le i \le m-1$ and
$P_m = (v_{m+1}, v_m, v_{2m+1}, v_{2m}, v_1)$.

Then the paths $P_1, P_2, \ldots, P_{m-1}$ and P_m form an m-graphoidal path cover ψ of W_n and so $\eta_m(W_n) \le |\psi| = m = n - (m+1)$ as $n = 2m+1$.
Hence $\eta_m(W_n) = n - (m+1)$ for $n = 2m+1$.

If $n \le 2m$, then $n - 1 \le 2m - 1$ and so $\Delta(W_n) \le 2m - 1$. Then from Corollary 2.15, it follows that $\eta_m(W_n) = n - m$.

Corollary 2.21. $\eta_{\lfloor \frac{n-1}{2} \rfloor}(W_n) = \pi(W_n)$.

Proof. From Theorem 2.20, it follows that

$$\begin{aligned} \eta_{\lfloor \frac{n-1}{2} \rfloor}(W_n) &= n - \left(\left\lfloor \frac{n-1}{2} \right\rfloor + 1 \right) \\ &= n - \left\lceil \frac{n}{2} \right\rceil \\ &= \left\lfloor \frac{n}{2} \right\rfloor \\ &= \pi(W_n). \end{aligned}$$

Theorem 2.22. For a complete graph K_p $(p \ge 4)$,

$$\eta_m(K_p) = \begin{cases} \dfrac{p(p-(2m+1))}{2}, & \text{if} \quad p \ge 2m+2 \\ \dfrac{p+1}{2}, & \text{if} \quad p = 2m+1 \\ \dfrac{p(p-(2m-1))}{2}, & \text{if} \quad p \le 2m. \end{cases}$$

Proof. Let $V(K_p) = \{v_1, v_2, \cdots, v_{p-1}, v_p\}$. Then $q = \dfrac{p(p-1)}{2}$.

Case(i) $p > 2m+1$.

If p is odd, then let $p = 2n+1$ and consider the paths
$P_1 = (v_1, v_{2n+1}, v_2, v_{2n}, v_3, v_{2n-1}, \cdots, v_{n+2}, v_{n+1})$

$P_2 = (v_2, v_1, v_3, v_{2n+1}, v_4, v_{2n}, \cdots, v_{n+3}, v_{n+2})$
$P_3 = (v_3, v_2, v_4, v_1, v_5, v_{2n+1}, \cdots, v_{n+4}, v_{n+3})$
$\cdots \quad \cdots \quad \cdots$
$P_m = (v_m, v_{m-1}, v_{m+1}, v_{m-2}, v_{m+2}, v_{m-3}, \cdots, v_{n+m+1}, v_{n+m})$ and
$P_{m+1} = (v_{m+1}, v_{n+1}, v_m, v_{n+2}, v_{m-1}, v_{n+3}, v_{m-2}, \cdots, v_{n+m-1}, v_2, v_{n+m}, v_1, v_{n+m+1})$
If p is even, then let $p = 2n$ and consider the paths
$P_1 = (v_1, v_{2n}, v_2, v_{2n-1}, v_3, v_{2n-2}, \cdots, v_n, v_{n+1})$
$P_2 = (v_2, v_1, v_3, v_{2n}, v_4, v_{2n-1}, \cdots, v_{n+1}, v_{n+2})$
$P_3 = (v_3, v_2, v_4, v_1, v_5, v_{2n}, \cdots, v_{n+2}, v_{n+3})$
$\cdots \quad \cdots \quad \cdots$
$P_m = (v_m, v_{m-1}, v_{m+1}, v_{m-2}, v_{m+2}, v_{m-3}, \ldots, v_{n+m-1}, v_{n+m})$ and
$P_{m+1} = (v_{m+1}, v_m, v_{n+1}, v_{m-1}, v_{n+2}, v_{m-2}, \ldots, v_1, v_{n+m}, v_{n+m+1})$

We see that the paths $P_1, P_2, \cdots, P_m, P_{m+1}$ form an m-graphoidal path cover of K_p in which every vertex is an internal vertex of exactly m paths. Then from Corollary 2.13, it follows that

$$\begin{aligned}\eta_m(K_p) &= q - mp \\ &= \frac{p(p-1)}{2} - mp \\ &= \frac{p(p-(2m+1))}{2}\end{aligned}$$

Case(ii) $p = 2m+1$.
Now, consider the paths
$P_1 = (v_{2m+1}, v_3, v_{2m}, v_4, v_{2m-1}, \cdots, v_m, v_{m+3}, v_{m+1}, v_{m+2}, v_1, v_2)$
$P_2 = (v_2, v_4, v_{2m+1}, v_5, v_{2m}, \cdots, v_{m+1}, v_{m+4}, v_{m+2}, v_{m+3}, v_1, v_3)$
$P_3 = (v_3, v_5, v_2, v_6, v_{2m+1}, \cdots, v_{m+2}, v_{m+5}, v_{m+3}, v_{m+4}, v_1, v_4)$
$\cdots \quad \cdots \quad \cdots$
$P_{m-1} = (v_{m-1}, v_{m+1}, v_{m-2}, v_{m+2}, v_{m-3}, \cdots, v_{2m-2}, v_{2m+1}, v_{2m-1}, v_{2m}, v_1, v_m)$
$P_m = (v_m, v_{m+2}, v_{m-1}, v_{m+3}, v_{m-2}, \cdots, v_{2m-1}, v_2, v_{2m}, v_{2m+1}, v_1, v_{m+1})$ and
$P_{m+1} = (v_{2m+1}, v_2, v_3, v_4, v_5, \cdots, v_{m-1}, v_m, v_{m+1})$.

The paths P_i $(1 \le i \le m)$ can be obtained from m hamiltonian cycles of K_{2m+1} by removing an edge from each cycle and the path P_{m+1} is obtained by joining the removed edges and so $\eta_m(K_p) \le m+1$. Also, $\eta_m(K_p) \ge \frac{q}{p-1} = \frac{p}{2} = \frac{2m+1}{2} = m + \frac{1}{2}$ and hence $\eta_m(K_p) = m+1 = \frac{p+1}{2}$.

If $p \le 2m$, then $p-1 \le 2m-1$ and so $\Delta(K_p) \le 2m-1$. Then from Corollary 2.15, it follows that $\eta_m(K_p) = \frac{p(p-(2m-1))}{2}$.

Corollary 2.23. $\eta_{\lfloor \frac{p-1}{2} \rfloor}(K_p) = \pi(K_p)$.

Proof. From Theorem 2.22, it follows that $\eta_{(\frac{p-1}{2})}(K_p) = \frac{p+1}{2}$, if p is odd and $\eta_{(\frac{p-2}{2})}(K_p) = \frac{p}{2}$, if p is even. Thus $\eta_{\lfloor \frac{p-1}{2} \rfloor}(K_p) =$

$\left\lceil \frac{p}{2} \right\rceil = \pi(K_p)$.

Next we find η_m for tree, unicyclic graph and star.

Theorem 2.24. For any tree T, there exists an m-graphoidal cover ψ of T such that every vertex of degree greater than one is an internal vertex of some ψ - path.

Proof. The proof is given by induction on p. We can assume that $p > 1$ and the conclusion holds for any tree whose number of vertices is less than p. Let u be a pendant vertex and let v be a vertex which is adjacent to u. Let ψ_1 be an m-graphoidal path cover of $T - u$, satisfying the conclusion. If v is a pendant vertex of $T - u$, then replacing the path P in ψ_1, which ends at v by the path $P \cup \{uv\}$, we get a m-graphoidal path cover of T. This satisfies the requirement; otherwise $\psi_1 \cup \{uv\}$ is the required m-graphoidal path cover of T.

Corollary 2.25. Let n be the number of pendant vertices of a tree T. Then $\eta_m(T) = n - 1 - \sum_{k=2}^{m}(k-1)t_k$

Proof. From Theorem 2.24, it follows that $t = n$.
By Theorem 2.8,

$$\begin{aligned}
\eta_m(T) &= q - p - \sum_{k=2}^{m}(k-1)t_k + t \\
&= p - 1 - p - \sum_{k=2}^{m}(k-1)t_k + n \\
&= n - 1 - \sum_{k=2}^{m}(k-1)t_k.
\end{aligned}$$

Corollary 2.26. For any tree T, $\eta_m(T) \geq \Delta - 1 - \sum_{k=2}^{m}(k-1)t_k$

Proof. T has at least Δ vertices of degree one and the result follows from Corollary 2.25.

Corollary 2.27. Let T be a tree with $\Delta \geq 2m + 1$. Let v be a vertex in T such that $d(v) = \Delta$. Then $\eta_m(T) = \Delta - m$ if $d(w) = 1$ or 2 for all other vertices $w \neq v$.

Proof. T has exactly Δ vertices of degree one if and only if $d(w) = 1$ or 2 for all other vertices $w \neq v$. Then $t_k = 0$ for $k = 2, 3, \cdots, m-1$ and $t_m = 1$ and $n = \Delta$. The result follows from Corollary 2.25.

Theorem 2.28. Let G be an unicycle graph with n pendant vertices. Let C be the unique cycle in G. Let l be the number of vertices of degree greater than 2 on C. Then

$$\eta_m(G) = \begin{cases} 2 & \text{if} \quad l = 0, \\ n + 1 - \sum_{k=2}^{m}(k-1)t_k & \text{if} \quad l = 1, \\ n - \sum_{k=2}^{m}(k-1)t_k & \text{if} \quad \textit{otherwise}. \end{cases}$$

Proof. Case (i) : $l = 0$. Then $G = C$ and $\eta_m(G) = 2$.
Case (ii) : $l = 1$.

Let v be the unique vertex of degree greater than 2 on C. Let $e = uv$ be an edge on C incident at v. Then $G-e$ is a tree with $n+1$ pendant vertices and it follows from Corollary 2.25 that $\eta_m(G-e) = n - \sum_{k=2}^{m}(k-1)t_k$.

Let ψ_1 be a minimum m-graphoidal path cover of $G-e$. Then $\psi_1 \cup \{P\}$, where P is a path of length one consisting of the edge e is a m-graphoidal path cover of G so that $\eta_m(G) \leq n+1-\sum_{k=2}^{m}(k-1)t_k$. Further for any m-graphoidal path cover ψ of G, all the n pendant vertices and at least one vertex on C are not internal in any path in ψ. Hence $t \geq n+1$ and by the Theorem 2.8, $\eta_m(G) \geq n+1-\sum_{k=2}^{m}(k-1)t_k$. Hence $\eta_m(G) = n+1-\sum_{k=2}^{m}(k-1)t_k$.

Case (iii) : $l > 1$.

Let v, w be vertices of degree greater than 2 on C such that all vertices in a (v,w) - section of C other than v, w have degree 2. Let P denote this (v,w)-section. If P has length 1, let $G_1 = G - e$ where e is the edge vw. Otherwise let G_1 be the subgraph obtained by deleting all the internal vertices of P. Clearly G_1 is a tree with n pendant vertices and hence $\eta_m(G_1) = n-1-\sum_{k=2}^{m}(k-1)t_k$. If ψ_1 is a minimum m-graphoidal path cover of G_1, then $\psi_1 \cup \{P\}$ is a m-graphoidal path cover of G and hence $\eta_m(G) \leq n-\sum_{k=2}^{m}(k-1)t_k$. Since G has n pendant vertices, $t \geq n$ and again by the Theorem 2.8, $\eta_m(G) \geq n-\sum_{k=2}^{m}(k-1)t_k$. Hence $\eta_m(G) = n-\sum_{k=2}^{m}(k-1)t_k$.

Theorem 2.29. For a star $K_{1,n}$ $(n \geq 2)$,

$$\eta_m(K_{1,n}) = \begin{cases} n-m & \text{if} \quad n \geq 2m \\ n-(m-1) & \text{if} \quad n \leq 2m-1. \end{cases}$$

Proof. Now, let $X = \{x_1\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$.
Suppose $n \geq 2m$. Then $d(x_1) \geq 2m$ and so $t_m \leq 1$. Also $t_k = 0$ for $k = 2, 3, \cdots, m-1$. Since $K_{1,n}$ is a tree with n pendant vertices, it follows from Corollary 2.26 that $\eta_m(K_{1,n}) \geq n-1-(m-1) = n-m$. Now, the paths $P_i = (y_{2i-1}, x_1, y_{2i})$ $(1 \leq i \leq m)$ together with the remaining edges form a m-graphoidal path cover ψ of G such that

$$\begin{aligned} \eta_m(K_{1,n}) \leq |\psi| &= m + |E(K_{1,n}) - \{E(P_1) \cup E(P_2) \ldots E(P_m)\}| \\ &= m + (n-2m) \\ &= n-m. \end{aligned}$$

Hence $\eta_m(K_{1,n}) = n-m$.

Now, suppose $n \leq 2m-1$ and so $\Delta(K_{1,n}) \leq 2m-1$. Then from Corollary 2.15, it follows that $\eta_m(K_{1,n}) = n-(m-1)$.

Corollary 2.30. $\eta_{\lfloor \frac{n}{2} \rfloor}(K_{1,n}) = \pi(K_{1,n})$.

Proof. From Theorem 2.29, it follows that
$\eta_{\lfloor \frac{n}{2} \rfloor}(K_{1,n}) = n - \left(\lfloor \frac{n}{2} \rfloor\right) = \lceil \frac{n}{2} \rceil = \pi(K_{1,n})$.

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On the cubic Gauss sums and its fourth power mean[1]

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Abstract The main purpose of this paper is to study the calculating problem of the fourth power mean of the cubic Gauss sums, and give an exact calculating formula for it.

Keywords Cubic Gauss sums, fourth power mean, calculating formula.

§1. Introduction

Let $q \geq 3$ be a positive integer. For any fixed positive integer r and integer n, we define the r-th Gauss sums $G(n, r, q)$ as follows:

$$G(n, r, q) = \sum_{a=1}^{q} e\left(\frac{na^r}{q}\right),$$

where $e(y) = e^{2\pi i y}$. This summation is more important, because it is a generalization of the quadratic Gauss sums $G(n; q) = \sum_{a=1}^{q} e\left(\frac{na^2}{q}\right)$. The various properties of $G(n; q)$ were investigated by many authors (see [1] and [2]). For any positive integer $r \geq 3$, using the elementary method we can easily prove that (see [3])

$$|G(n, r, q)| \leq \sqrt{q} \prod_{p^\alpha \| q} (r, \phi(p^\alpha)),$$

where $\phi(q)$ is the Euler function, n is any integer with $(n, q) = 1$, and $\prod_{p^\alpha \| q}$ denotes the product over all prime divisors of q with $p^\alpha | q$ and $p^\alpha \nmid q$.

But about the other properties of $G(n, r, q)$, we know very little at present. For $r = 2$, we have studied its properties and given two accurate calculation formulas for the k-th power mean of this sum(see [4]).

The main purpose of this paper is to study the fourth power mean properties of $\sum_{n=1}^{q} |$ $G(n, 3, q)$ $|^4$, and give an accurate calculating formula for it. That is, we shall prove the following main conclusion:

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Theorem. Let $q = 3^\beta q_1$ be a positive integer, where $\beta \geq 0$ and $3 \nmid q_1$. Then we have the calculating formula

$$\sum_{a=1}^{q}\left|\sum_{b=1}^{q} e\left(\frac{ab^3}{q}\right)\right|^4 = q^4\left[3 - \frac{7-(-1)^{\left[\frac{\beta+1}{3}\right]-\left[\frac{\beta}{3}\right]}}{3^{1+\left[\frac{\beta+1}{3}\right]}}\right]$$
$$\times \prod_{\substack{p^\alpha \| q\\ p\equiv 1(3)}}\left[1+\left(\frac{7}{p}+\frac{1}{p^2}\right)\left(1-\frac{1}{p^{\left[\frac{\alpha+2}{3}\right]}}\right)-\frac{1}{p^{\left[\frac{\alpha+2}{3}\right]}}\left(1-\frac{1}{p^{3\left[\frac{\alpha+2}{3}\right]-\alpha}}\right)\right],$$

where $\prod\limits_{p^\alpha \| q}$ denotes the product over all p such that $p^\alpha \mid q$ and $p^{\alpha+1} \nmid q$.

For general positive integer $m \geq 3$ and r with $r \mid q-1$, whether there exists an calculating formula for $\sum\limits_{n=1}^{q} |G(n,r,q)|^{2m}$ is an open problem.

§2. Some Lemmas

To complete the proof of the theorem, we need the following Lemmas.

Lemma 1. Let p be a prime with $p \equiv 1(\mathrm{mod}3)$. Then we have the identity

$$\sum_{a=1}^{p^\alpha}{}'\left|\sum_{b=1}^{p^\alpha} e\left(\frac{ab^3}{p^\alpha}\right)\right|^4 = \begin{cases} p^{8k}\phi(p^{3k}), & \text{if } \alpha = 3k;\\ 6p^{8k+2}\phi(p^{3k+1}), & \text{if } \alpha = 3k+1;\\ p^{8k+4}\phi(p^{3k+2}), & \text{if } \alpha = 3k+2,\end{cases}$$

where $\phi(d)$ is the Euler function.

Proof. Let $\chi_3(b)$ be a cubic character modulo p, then we have

$$\begin{aligned}\sum_{a=1}^{p}{}'\left|\sum_{b=1}^{p} e\left(\frac{ab^3}{p}\right)\right|^4 &= \sum_{a=1}^{p-1}\left|1+\sum_{b=1}^{p-1}\left(1+\chi_3(b)+\chi_3^2(b)\right)e\left(\frac{ab}{p}\right)\right|^4 \qquad (1)\\ &= \sum_{a=1}^{p-1}\left|1+\sum_{b=1}^{p-1} e\left(\frac{ab}{p}\right)+\sum_{b=1}^{p-1}\chi_3(b)e\left(\frac{ab}{p}\right)+\sum_{b=1}^{p-1}\chi_3^2(b)e\left(\frac{ab}{p}\right)\right|^4\\ &= \sum_{a=1}^{p-1}\left|\overline{\chi}_3(a)\tau(\chi_3)+\overline{\chi}_3^2(a)\tau(\chi_3^2)\right|^4\\ &= \sum_{a=1}^{p-1}\left|\overline{\chi}_3(a)\tau(\chi_3)+\chi_3(a)\tau(\overline{\chi_3})\right|^4\\ &= 6(p-1)p^2.\end{aligned}$$

If $\alpha > 1$ and $(a,p) = 1$(see reference [4]), then we have

$$\sum_{b=1}^{p^\alpha}{}'\left(1+\chi_3(b)+\chi_3^2(b)\right)e\left(\frac{ab}{p^\alpha}\right) = \mu(p^\alpha)+\overline{\chi}_3(a)\tau(\chi_3)+\overline{\chi}_3^2(a)\tau(\chi_3^2) = 0. \qquad (2)$$

Let $S_1(p^\alpha)=\sum_{a=1}^{p^\alpha}{}' \left|\sum_{b=1}^{p^\alpha} e\left(\frac{ab^3}{p^\alpha}\right)\right|^4$, then from (2) we obtain

$$S_1(p^2)=\sum_{a=1}^{p^2}{}' \left|\sum_{b=1}^{p^2} e\left(\frac{ab^3}{p^2}\right)\right|^4=\sum_{a=1}^{p^2}{}' \left|\sum_{b=1}^{p^2}{}' e\left(\frac{ab^3}{p^2}\right)+p\right|^4=p^4\phi(p^2), \tag{3}$$

$$S_1(p^3)=\sum_{a=1}^{p^3}{}' \left|\sum_{b=1}^{p^3} e\left(\frac{ab^3}{p^3}\right)\right|^4=\sum_{a=1}^{p^3}{}' \left|\sum_{b=1}^{p^3}{}' e\left(\frac{ab^3}{p^3}\right)+p^2\right|^4=p^8\phi(p^3), \tag{4}$$

and

$$\begin{aligned} S_1(p^\alpha) \quad &=\sum_{a=1}^{p^\alpha}{}' \left|\sum_{b=1}^{p^\alpha} e\left(\frac{ab^3}{p^\alpha}\right)\right|^4=\sum_{a=1}^{p^\alpha}{}' \left|\sum_{b=1}^{p^\alpha}{}' e\left(\frac{ab^3}{p^\alpha}\right)+\sum_{b=1}^{p^{\alpha-1}} e\left(\frac{ab^3}{p^{\alpha-3}}\right)\right|^4 \\ &=\sum_{a=1}^{p^\alpha}{}' \left|\sum_{b=1}^{p^\alpha}{}' \left(1+\chi_3(b)+\chi_3^2(b)\right) e\left(\frac{ab}{p^\alpha}\right)+p^2\sum_{b=1}^{p^{\alpha-3}} e\left(\frac{ab^3}{p^{\alpha-3}}\right)\right|^4 \\ &=p^{11}\sum_{a=1}^{p^{\alpha-3}}{}' \left|\sum_{b=1}^{p^{\alpha-3}} e\left(\frac{ab^3}{p^{\alpha-3}}\right)\right|^4 \\ &=p^{11}S_1(p^{\alpha-3}). \end{aligned} \tag{5}$$

If $\alpha=3k$, combining (4) and (5) we have

$$S_1(p^\alpha)=p^{11}S_1(p^{\alpha-3})=p^{22}S_1(p^{\alpha-6})=\cdots=p^{11(k-1)}S_1(p^3)=p^{8k}\phi(p^{3k}).$$

If $\alpha=3k+1$, combining (1) and (5) we have

$$S_1(p^\alpha)=p^{11}S_1(p^{\alpha-3})=p^{22}S_1(p^{\alpha-6})=\cdots=p^{11k}S_1(p)=6p^{8k+2}\phi(p^{3k+1}).$$

If $\alpha=3k+2$, combining (3) and (5) we have

$$S_1(p^\alpha)=p^{11}S_1(p^{\alpha-3})=p^{22}S_1(p^{\alpha-6})=\cdots=p^{11k}S_1(p^2)=p^{8k+4}\phi(p^{3k+2}).$$

This completes the proof of Lemma 1.

Lemma 2. Let p be a prime with $p\equiv 1(\mathrm{mod}3)$. Then we have the identity

$$\sum_{a=1}^{p^\alpha}\left|\sum_{b=1}^{p^\alpha} e\left(\frac{ab^3}{p^\alpha}\right)\right|^4=p^{4\alpha}\left[1+\left(\frac{7}{p}+\frac{1}{p^2}\right)\left(1-\frac{1}{p^{\left[\frac{\alpha+2}{3}\right]}}\right)-\frac{1}{p^{\left[\frac{\alpha+2}{3}\right]}}\left(1-\frac{1}{p^{3\left[\frac{\alpha+2}{3}\right]-\alpha}}\right)\right].$$

Proof. Let $S(p^\alpha) = \sum_{a=1}^{p^\alpha}\left|\sum_{b=1}^{p^\alpha} e\left(\frac{ab^3}{p^\alpha}\right)\right|^4$, we have

$$\begin{aligned}S(p^\alpha) &= \sideset{}{'}\sum_{a=1}^{p^\alpha}\left|\sum_{b=1}^{p^\alpha} e\left(\frac{ab^3}{p^\alpha}\right)\right|^4 + \sum_{a=1}^{p^{\alpha-1}}\left|\sum_{b=1}^{p^\alpha} e\left(\frac{ab^3}{p^{\alpha-1}}\right)\right|^4 \qquad (6)\\ &= S_1(p^\alpha) + p^4\sum_{a=1}^{p^{\alpha-1}}\left|\sum_{b=1}^{p^{\alpha-1}} e\left(\frac{ab^2}{p^{\alpha-1}}\right)\right|^4\\ &= S_1(p^\alpha) + p^4 S(p^{\alpha-1})\\ &= S_1(p^\alpha) + p^4 S_1(p^{\alpha-1}) + p^8 S(p^{\alpha-2})\\ &= S_1(p^\alpha) + p^4 S_1(p^{\alpha-1}) + p^8 S_1(p^{\alpha-2}) + \cdots + p^{4(\alpha-2)}S_1(p^2) + p^{4(\alpha-1)}S(p).\end{aligned}$$

If $\alpha = 3k$, note that $S(p) = \sum_{a=1}^{p}\left|\sum_{b=1}^{p} e\left(\frac{ab^3}{p}\right)\right|^4 = p^4 + S_1(p) = p^4 + 6p^2(p-1)$, (6) and Lemma 1 we have

$$\begin{aligned}&S(p^\alpha) = \phi(p^\alpha)p^{\frac{8}{3}\alpha} + p^4\phi(p^{\alpha-1})p^{\frac{8}{3}(\alpha-1)-\frac{4}{3}} + 6p^8\phi(p^{\alpha-2})p^{\frac{8}{3}(\alpha-2)-\frac{2}{3}}\\ &+\cdots + p^{4(\alpha-2)}p^4\phi(p^2) + p^{4(\alpha-1)}\left(p^4 + 6p^2(p-1)\right)\\ &= p^{4\alpha}\left[7\left(\frac{1}{p^2}+\frac{1}{p^3}+\cdots+\frac{1}{p^{\frac{\alpha}{3}+1}}\right) + \left(\frac{1}{p^3}+\frac{1}{p^4}+\cdots+\frac{1}{p^{\frac{\alpha}{3}+2}}\right)\right](p-1) + p^{4\alpha}\\ &= p^{4\alpha}\left[1 + \left(\frac{7}{p}+\frac{1}{p^2}\right)\left(1-\frac{1}{p^{\frac{\alpha}{3}}}\right)\right].\end{aligned}$$

Applying the same methods, we also have

$$S(p^\alpha) = p^{4\alpha}\left[1 + \left(\frac{7}{p}+\frac{1}{p^2}\right)\left(1-\frac{1}{p^{\frac{\alpha+1}{3}}}\right) - \frac{1}{p^{\frac{\alpha+1}{3}}}\left(1-\frac{1}{p}\right)\right], \quad \text{if } \alpha = 3k+2;$$

and

$$S(p^\alpha) = p^{4\alpha}\left[1 + \left(\frac{7}{p}+\frac{1}{p^2}\right)\left(1-\frac{1}{p^{\frac{\alpha+2}{3}}}\right) - \frac{1}{p^{\frac{\alpha+2}{3}}}\left(1-\frac{1}{p^2}\right)\right], \quad \text{if } \alpha = 3k+1.$$

This completes the proof of Lemma 2.

Lemma 3. For any integer $\alpha \geq 0$, we have the identity

$$\sum_{a=1}^{3^\alpha}\left|\sum_{b=1}^{3^\alpha} e\left(\frac{ab^3}{3^\alpha}\right)\right|^4 = 3^{4\alpha}\left(3 - \frac{7-(-1)^{\left[\frac{\alpha+1}{3}\right]-\left[\frac{\alpha}{3}\right]}}{3^{1+\left[\frac{\alpha+1}{3}\right]}}\right).$$

Proof. It is clear that Lemma 3 holds if $\alpha = 0$. So we suppose that $\alpha \geq 1$. Note that $\sum_{a=1}^{2} e\left(\frac{a}{3}\right) = -1$, we have

$$\begin{aligned} S_1(3^2) &= \sum_{a=1}^{3^2}{}' \left| \sum_{b=1}^{3^2} e\left(\frac{ab^3}{3^2}\right) \right|^4 = \sum_{a=1}^{3^2}{}' \left| \sum_{b=1}^{3^2}{}' e\left(\frac{ab^3}{3^2}\right) + 3 \right|^4 \\ &= \sum_{a=1}^{3^2}{}' \left| 3e\left(\frac{a}{9}\right) + 3e\left(\frac{-a}{9}\right) + 3 \right|^4 = 3^4 \sum_{a=1}^{3^2}{}' \left| e\left(\frac{a}{9}\right) + e\left(\frac{-a}{9}\right) + 1 \right|^4 \\ &= 3^4\phi(3^2)(6+12+1) + 3^4 \sum_{a=1}^{3^2}{}' 4\left(e\left(\frac{a}{3}\right) + e\left(\frac{-a}{3}\right)\right) \\ &= 19 \cdot 3^4\phi(3^2) + 3^4 \cdot 24 \sum_{a=1}^{2} e\left(\frac{a}{3}\right) \\ &= 15 \cdot 3^4\phi(3^2). \end{aligned} \tag{7}$$

Applying the method of proving (5), we may get

$$S_1(3^\alpha) = \sum_{a=1}^{3^\alpha}{}' \left| \sum_{b=1}^{3^\alpha} e\left(\frac{ab^3}{3^\alpha}\right) \right|^4 = 3^{11} S_1(3^{\alpha-3}). \tag{8}$$

Note that $S_1(3) = \sum_{a=1}^{3}{}' \left| \sum_{b=1}^{3} e\left(\frac{ab^3}{3}\right) \right|^4 = 0$ and (8), we have

$$S_1(3^\alpha) = 3^{11k} S_1(3) = 0, \quad \text{if} \ \ \alpha = 3k+1.$$

If $\alpha = 3k$, from (8) we have

$$\begin{aligned} S_1(3^\alpha) &= 3^{11(k-1)} S_1(3^3) = 3^{11(k-1)} \sum_{a=1}^{3^3}{}' \left| \sum_{b=1}^{3^3} e\left(\frac{ab^3}{3^3}\right) \right|^4 \\ &= 3^{11(k-1)} \sum_{a=1}^{3^3}{}' \left| \sum_{b=1}^{3^3}{}' e\left(\frac{ab^3}{3^3}\right) + 3^2 \right|^4 = 3^{11(k-1)} \cdot 3^8\phi(3^3) = 3^{8k}\phi(3^{3k}). \end{aligned}$$

If $\alpha = 3k+2$, combining (7) and (8) we have

$$S_1(3^\alpha) = 3^{11k} S_1(3^2) = 15 \cdot 3^{8k+4}\phi(3^{3k+2}).$$

Applying the method of proving Lemma 2, we have

$$\begin{aligned} & S(3^\alpha) = S_1(3^\alpha) + 3^4 S_1(3^{\alpha-1}) + 3^8 S_1(3^{\alpha-2}) + \cdots + 3^{4(\alpha-2)} S_1(3^2) + 3^{4(\alpha-1)} S(3) \\ &= \phi(3^\alpha) 3^{\frac{8}{3}\alpha} + 15 \cdot 3^4 \phi(3^{\alpha-1}) 3^{\frac{8}{3}(\alpha-1) - \frac{4}{3}} + 0 + \cdots + 15 \cdot 3^{4(\alpha-2)} 3^4 \phi(3^2) + 3^{4(\alpha-1)} 3^4 \\ &= 3^{4\alpha} \left[\left(\frac{1}{3^2} + \frac{1}{3^3} + \cdots + \frac{1}{3^{\frac{\alpha}{3}+1}} \right) + 15 \left(\frac{1}{3^3} + \frac{1}{3^4} + \cdots + \frac{1}{3^{\frac{\alpha}{3}+2}} \right) \right] (3-1) + 3^{4\alpha} \\ &= 3^{4\alpha} \left[1 + \left(\frac{1}{3} + \frac{15}{3^2} \right) \left(1 - \frac{1}{3^{\frac{\alpha}{3}}} \right) \right] \\ &= 3^{4\alpha} \left[3 - \frac{1}{3^{\frac{\alpha}{3}}} \right], \quad \text{if} \ \ \alpha = 3k. \end{aligned}$$

Similarly, we may obtain $S(3^\alpha) = 3^{4\alpha}\left(3 - \dfrac{7-(-1)^{\left[\frac{\alpha+1}{3}\right]-\left[\frac{\alpha}{3}\right]}}{3^{1+\left[\frac{\alpha+1}{3}\right]}}\right)$. This proves Lemma 3.

§3. Proof of the theorem

From Lemma 2 and Lemma 3 on the above section, we can complete the proof of the theorem. Let q has the prime power decomposition $q = \prod_{i=1}^{r} p_i^{\alpha_i}$. It is clear that if a_i pass through a complete residue system modulo $p_i^{\alpha_i}$, then $a = \sum_{i=1}^{r} a_i \frac{q}{p_i^{\alpha_i}} \equiv \sum_{i=1}^{r} a_i M_i$ pass through a complete residue system modulo q. Note that $\sum_{a=1}^{p^\alpha}\left|\sum_{b=1}^{p^\alpha} e\left(\frac{ab^3}{p^\alpha}\right)\right|^4 = \sum_{a=1}^{p^\alpha}\left|\sum_{b=1}^{p^\alpha} e\left(\frac{ab}{p^\alpha}\right)\right|^4 = p^{4\alpha}$, if $p \equiv 2(\mathrm{mod} 3)$ and the multiplicative property of $\left|\sum_{b=1}^{q} e\left(\frac{ab^3}{q}\right)\right|^4$, from Lemma 2 and Lemma 3 we may immediately obtain the identity

$$\begin{aligned}
\sum_{a=1}^{q}\left|\sum_{b=1}^{q} e\left(\frac{ab^3}{q}\right)\right|^4 &= \prod_{i=1}^{r}\left[\sum_{a_i=1}^{p_i^{\alpha_i}}\left|\sum_{b_i=1}^{p_i^{\alpha_i}} e\left(\frac{a_i M_i^3 b_i^3}{p_i^{\alpha_i}}\right)\right|^k\right] \\
&= \prod_{\substack{i=1 \\ p_i\equiv 1(3)}}^{r}\left[p_i^{4\alpha_i}\left(1+\left(\frac{7}{p_i}+\frac{1}{p_i^2}\right)\left(1-\frac{1}{p_i^{\left[\frac{\alpha_i+2}{3}\right]}}\right)-\frac{1}{p_i^{\left[\frac{\alpha_i+2}{3}\right]}}\left(1-\frac{1}{p_i^{3\left[\frac{\alpha_i+2}{3}\right]-\alpha_i}}\right)\right)\right] \\
&\times \prod_{\substack{i=1 \\ p_i\equiv 2(3)}}^{r} p_i^{4\alpha_i} \times 3^{4\beta}\left(3-\frac{7-(-1)^{\left[\frac{\beta+1}{3}\right]-\left[\frac{\beta}{3}\right]}}{3^{1+\left[\frac{\beta+1}{3}\right]}}\right) \\
&= q^4\left[3-\frac{7-(-1)^{\left[\frac{\beta+1}{3}\right]-\left[\frac{\beta}{3}\right]}}{3^{1+\left[\frac{\beta+1}{3}\right]}}\right] \\
&\times \prod_{\substack{p^\alpha \| q \\ p\equiv 1(3)}}\left[1+\left(\frac{7}{p}+\frac{1}{p^2}\right)\left(1-\frac{1}{p^{\left[\frac{\alpha+2}{3}\right]}}\right)-\frac{1}{p^{\left[\frac{\alpha+2}{3}\right]}}\left(1-\frac{1}{p^{3\left[\frac{\alpha+2}{3}\right]-\alpha}}\right)\right].
\end{aligned}$$

This completes the proof of Theorem.

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On the dual functions $Z_*(n)$ and $S_*(n)$

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Abstract The pseudo Smarandache dual function, denoted by $Z_*(n)$, is defined as the maximum positive integer m such that $\frac{m(m+1)}{2}$ divides n. The Smarandache dual function, denoted by $S_*(n)$, is defined as the maximum positive integer m such that m! divides n. This paper derives the explicit expressions for $Z_*(2p^k), Z_*(3p^k), Z_*(4p^k)$ and $Z_*(5p^k)$, where p is an odd prime, as well as an inequality involving $S_*((2n+1)!(2n+3)!)$.

Keywords Pseudo Smarandache dual function, Smarandache dual function.

§1. Introduction

The pseudo Smarandache dual function, denoted by $Z_*(n)$, introduced by Sandor [1], is defined as follows (where $\mathbb{Z}^+$ is the set of all positive integers).

Definition 1.1. For any integer $n \geq 1$, $Z_*(n) = \max\left\{m \ : \ \text{m} \in \mathbb{Z}^+, \ \frac{m(m+1)}{2} \,|n\right\}$.

Sandor [1] has also studied some elementary properties of the function $Z_*(n)$. They are given in the following lemmas.

Lemma 1.1. For any integer $n \geq 1$, $1 \leqslant Z_*(n) \leqslant \frac{1}{2}(\sqrt{1+8n} - 1)$.

Lemma 1.2. $Z_*(\frac{k(k+1)}{2}) = k$ for any integer $k \geq 1$.

Lemma 1.3. For any integers $a, b \geq 1$, $Z_*(ab) \geq \max\{Z_*(a), Z_*(b)\}$.

Lemma 1.4. Let $p \geq 3$ be a prime. Then, for any integer $k \geq 1$,

$$Z_*(p^k) = \begin{cases} 2, \text{ if } \ p = 3 \\ 1, \text{ if } p \neq 3 \end{cases}$$

Lemma 1.5. Any solution of the equation $Z(n) = Z_*(n)$ is of the form

$$n = \frac{k(k+1)}{2}, \ k \geq 1.$$

The Smarandache dual function, denoted by $S_*(n)$, has been defined by Sandor [2] as follows.

Definition 1.2. For any integer $n \geq 1$, $S_*(n) = \max\{m \ : \ \text{m} \in \mathbb{Z}^+, \ \text{m}! \,|n\}$.

Some elementary properties of the function $S_*(n)$, studied by Sandor [2], are given in the following lemmas.

Lemma 1.6. For any prime $p \geq 2$, and any integer $n \geq p$, $S_*(n! + (p-1)!) = p - 1$.

Lemma 1.7. For any integer $n \geq 1$,

$$S_*((2n)!(2n+2)!) \begin{cases} = 2n+2, & \text{if } 2n+3 \text{ is a prime} \\ \geq 2n+3, & \text{if } 2n+3 \text{ is not a prime} \end{cases}$$

It may be mentioned here that, there is some mistake in Proposition 8 in Sandor [2]; the correct form is given in Lemma 1.7 above.

In this paper, we derive the expressions for $Z_*(2p^k), Z_*(3p^k), Z_*(4p^k)$and $Z_*(5p^k)$. These are given in the next Section 2. In Section 3, we give a new inequality involving $S_*(n)$. Some concluding remarks are given in the final Section 4.

§2. The pseudo Smarandache dual function $Z_*(n)$

In this section, we derive the explicit expressions for $Z_*(2p^k), Z_*(3p^k), Z_*(4p^k)$ and $Z_*(5p^k)$, where p is a prime and $k \geq 1$ is an integer. They are given in the following lemmas.

In what follows, we shall denote by Tm the $m-th$ triangular number, that is,

$$T_m \equiv \frac{m(m+1)}{2}; \ m = 1, \ 2, \ \cdots .$$

Note that T_m is strictly increasing in m.

Lemma 2.1. Let $p \geq 3$ be a prime. Then, for any integer $k \geq 1$,

$$Z_*(2p^k) = \begin{cases} 3, & \text{if } p = 3 \\ 4, & \text{if } p = 5, \\ 1, & \text{if } p \geqslant 7, \end{cases}$$

Proof. By definition,

$$Z_*(2p^k) = \max\left\{m \ : \ \mathrm{m} \in \mathbb{Z}^+, \ \mathrm{T_m} \left| 2p^k \right. \right\} = \max\left\{m \ : \ \mathrm{m} \in \mathbb{Z}^+, \ \mathrm{m(m+1)} \left| 4p^k \right. \right\}.$$

Then, one of m and $m+1$ is p, and the other one must be 4. Now,

$$m+1 = 4, m = p \Longrightarrow p = 3, \qquad m+1 = p, m = 4 \Longrightarrow p = 5.$$

Thus, if $p \geq 7$, we must have $m = 1$.

Lemma 2.2. Let $p \geq 5$ be a prime. Then, for any integer $k \geq 1$,

$$Z_*(3p^k) = \begin{cases} 5, & \text{if p} = 5 \\ 6, & \text{if p} = 7 \\ 2, & \text{if p} \geqslant 11 \end{cases}$$

Proof. In this case, by definition,

$$Z_*(3p^k) = \max\left\{m \ : \ \mathrm{m} \in \mathbb{Z}^+, \ \mathrm{T_m} \left| 3p^k \right. \right\} = \max\left\{m \ : \ \mathrm{m} \in \mathbb{Z}^+, \ \mathrm{m(m+1)} \left| 6p^k \right. \right\}.$$

Now,

$$m+1=6, m=p \Longrightarrow p=5, \qquad m+1=p, m=6 \Longrightarrow p=7.$$

If $p \geq 11$, since T_2 divides 3 and $T_3 > 3$, it follows that $m=2$.

Lemma 2.3. Let $p \geq 3$ be a prime. Then, for any integer $k \geq 1$,

$$Z_*(4p^k) = \begin{cases} 7, & \text{if } p=7 \\ 1, & \text{otherwise} \end{cases}$$

Proof. Here,

$$Z_*(4p^k) = \max\left\{m \ : \ \mathrm{m} \in \mathbb{Z}^+, \ \mathrm{T_m} \left|4p^k\right.\right\} = \max\left\{m \ : \ \mathrm{m} \in \mathbb{Z}^+, \ \mathrm{m(m+1)} \left|8p^k\right.\right\}.$$

Now,

$$m+1=8, m=p \Longrightarrow p=7, \qquad m+1=p, m=8 \Longrightarrow p=9.$$

Thus, the second case cannot occur, and for $p \neq 7$, we must have $m=1$.

Lemma 2.4. Let $p \neq 5$ be a prime. Then,

$$Z_*(5k) = \begin{cases} 10, & \text{if } p=11 \\ 1, & \text{otherwise} \end{cases}$$

Proof. In this case, since

$$Z_*(5p) = \max\left\{m \ : \ \mathrm{m} \in \mathbb{Z}^+, \ \mathrm{T_m} \left|5p\right.\right\} = \max\left\{m \ : \ \mathrm{m} \in \mathbb{Z}^+, \ \mathrm{m(m+1)} \left|10p\right.\right\},$$

and since,

$$m+1=10, m=p \Longrightarrow p=9, \qquad m+1=p, m=10 \Longrightarrow p=11,$$

it follows that the first case cannot occur. Thus, for $p \neq 11$, we must have $m=1$.

Lemma 2.5. Let $p \neq 5$ be a prime. Then, for any integer $k \geq 2$,

$$Z_*(5p^k) = \begin{cases} p^2, & \text{if p} = 3 \\ 10, & \text{if p} = 11 \\ 1, & \text{otherwise} \end{cases}$$

Proof. Here,

$$Z_*(5p^k) = \max\left\{m \ : \ \mathrm{m} \in \mathbb{Z}^+, \ \mathrm{T_m} \left|5p^k\right.\right\} = \max\left\{m \ : \ \mathrm{m} \in \mathbb{Z}^+, \ \mathrm{m(m+1)} \left|10p^k\right.\right\}$$

When $p=3$, then $p^2+1=10$ divides 10. Therefore, in this case, $m=p^2$. Now,

$$m+1=10, m=p \Longrightarrow p=9, \qquad m+1=p, m=10 \Longrightarrow p=11.$$

Thus, the first case cannot occur.

§3. The Smarandache dual function $S_*(n)$

In this section, we prove the following result.

Lemma 3.1. For any integer $n \geq 1$, $S_*((2n+1)!(2n+3)!) \geq 2(n+2)$.

Proof. We first prove that

$2(n+2)$ divides $(2n+1)!$ for any integer $n \geq 1$.

The proof is as follows : If $n \geq 2$, then $2n+1 \geq n+2$ and so

$2(n+2)$ divides $(2n+1)! = (n+2)!(n+3)\cdots(2n).(2n+1)$.

Since 2×3 divides 3!, the result is true for $n = 1$ as well.

Now,

$$2(n+2) \ |(2\mathrm{n}+1)! \ \Rightarrow \ [2(\mathrm{n}+2)]! = (2n+3)! \ [2(\mathrm{n}+2)] \ |(2n+1)! \ (2n+3)!$$

This proves the lemma.

§4. Some remarks

The values of $S_*((2n+1)!(2n+3)!)$ for some small n are given in Sandor [2]. Motivated by these values, he makes the following conjecture :

Conjecture : $S_*((2n+1)!(2n+3)!) = q_n - 1$, where q_n is the first prime following $2n+3$.

It may be mentioned that the case $n = 2$ is a violation to the conjecture, since $S_*(5!7!) = 8$, though other values of n supports the conjecture. Le [3] claims to have proved the conjecture, using the inequality

$$ord_p((2n+1)!) + ord_p((2n+3)!) < ordp((q_n - 1)!),$$

where

$$ord_p(n!) = \sum_{k=1}^{\infty} \left[\frac{n}{p^k}\right] \text{ ([x] being the Gauss function of x),}$$

but the inequality remains to be proved.

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Smarandache sequence of Ulam numbers

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Abstract In this article, we present the results of investigation of Smarandache Concatenate Sequence formed from the sequence of Ulam Numbers, Ulam primes and report some primes and other results found from the sequence.

Keywords Ulam numbers, U-sequence, Smarandache U-sequence, reversed Smarandache U-sequence, prime, Ulam Prime numbers, UP-sequence, Smarandache UP-sequence.

§1. Introduction

The standard Ulam sequence starts with $U_1 = 1$ and $U_2 = 2$ being the first two Ulam numbers. Then for $n > 2$, U_n is defined to be the smallest integer that is the sum of two distinct earlier terms in exactly one way [1]. So, for example, 11 is an Ulam number because it is the sum of the pair of smaller Ulam numbers 8 and 3, and no other pair, while 13 is also an Ulam number because it is the sum of 11 and 2, and no other pair. 12 is not an Ulam number because it is the sum of 1 and 11, and of 4 and 8. The first few terms of sequence of Ulam numbers [3] are:

1, 2, 3, 4, 6, 8, 11, 13, 16, 18, 26, 28, 36, 38, 47, 48, 53, 57, 62, 69, 72, 77, 82, 87, 97, 99, 102, 106, 114, 126, 131, 138, 145, 148, 155, 175, 177, 180, 182, 189, 197, 206, 209, 219, 221, 236, 238, 241, 243, 253, 258, 260, 273, 282, 309, 316, 319, 324, 339, 341, 356, 358, 363, 370, 382, 390, 400, 402, 409, 412, 414, 429, 431, 434, 441, 451, 456, 483, 485, 497, 502, 522, 524, 544, 546, 566, 568, 585, 602, 605, 607, 612, 624, 627, 646, 668, 673, 685, 688, 690, 695, 720, 722, 732, 734, 739, 751, 781, 783, 798, 800, 820, 847, 849, 861, 864, 866, 891, 893, 905, 927, 949, 983, 986, 991, $\cdots$

Let us denote the sequence of Ulam numbers as U-sequence. So, the sequence of Ulam numbers,

$$U = 1, 2, 3, 4, 6, 8, 11, 13, 16, 18, 26, 28, 36, 38, 47, 48, 53, 57, \cdots$$

The Ulam numbers that are also prime numbers can be termed as Ulam Prime numbers. The first few terms of sequence of Ulam Prime numbers[4] are:

2, 3, 11, 13, 47, 53, 97, 131, 197, 241, 409, 431, 607, 673, 739, 751, 983, 991, 1103, 1433, 1489, 1531, 1553, 1709, 1721, 2371, 2393, 2447, 2633, 2789, 2833, 2897, 3041, 3109, 3217, 3371, 3373, 3527, 3547, 3593, 3671, 3691, 4057, 4153, 4211, 4297, 4363, 4409, 4451, 4517, 4519, 4729,

4903, 4969, 5059, 5081, 5531, 6029, 6481, 6569, 6833, 6911, 7043, 7219, 7297, 7459, 7537, 7559, 7583, 7603, 7691, 7727, 8011, 8101, 8167, 8539, 8573, 8969, 8971, 9013, 9137, 9311, 9377, 9511, 9619, 9643, 9721, 9743, 9851, 9941, $\cdots$

Let us denote the sequence of Ulam Prime numbers as UP-sequence. So, the sequence of Ulam Prime numbers,

$$UP = 2, 3, 11, 13, 47, 53, 97, 131, 197, 241, 409, 431, 607, 673, \cdots.$$

§2. Smarandache sequence

Let $S_1, S_2, S_3, \cdots, S_n, \cdots$ be an infinite integer sequence (termed as S-sequence), then the Smarandache sequence [5] or Smarandache Concatenated sequence [2] or Smarandache S-sequence is given by

$$S_1, \overline{S_1S_2}, \overline{S_1S_2S_3}, \cdots, \overline{S_1S_2S_3\cdots S_n}, \cdots$$

Also Smarandache Back Concatenated sequence or Reversed Smarandache S-sequence is

$$S_1, \overline{S_2S_1}, \overline{S_3S_2S_1}, \cdots, \overline{S_n\cdots S_3S_3S_1}, \cdots$$

§3. Smarandache U-Sequence

Smarandache sequence of Ulam numbers or Smarandache U-sequence is the sequence formed from concatenation of numbers in U-sequence (Note that U-sequence is the sequence of Ulam numbers). So, Smarandache U-sequence is
1, 12, 123, 1234, 12346, 123468, 12346811, 1234681113, 123468111316, 12346811131618, 1234681113161826, 123468111316182628, 12346811131618262836, 1234681113161826283638, $\cdots$
Let us denote the nth term of the Smarandache U-sequence by SU(n). So,

$$SU(1) = 1$$

$$SU(2) = 12$$

$$SU(3) = 123$$

$$SU(4) = 1234$$

$$SU(5) = 12346 \text{ and so on.}$$

§4. Observations on Smarandache U-sequence

We have investigated Smarandache U-sequence for the following two problems

(i) How many terms of Smarandache U-sequence are primes?

(ii) How many terms of Smarandache U-sequence belong to the initial U-sequence?

In search of answer to these problems, we find that
(a) There are only 2 primes in the first 3200 terms of Smarandache U-sequence. These are

$$SU(22) = 12346811131618262836384748535762697277.$$

$$\begin{aligned} SU(237) = \ & 12346811131618262836384748535762697277828797991021061141261311381451 \\ & 481551751771801821891972062092192212362382412432532582602732823093 \\ & 16319324339341356358363370382390400402409412414429431434444145145648 \\ & 3485497502522524544546566568585602605607612624627646668673685688690 \\ & 6957207227327347397517817837988008208478498618648668918939059279499 \\ & 8398699110181020102310251030103210351037105210791081110111031125115 \\ & 5115711641167116911861191120812301252125712961308131113131335133813 \\ & 4013551360137713871389140414061428143114331462146514701472148914921 \\ & 5091514151615311536153815501553159416021604161616411643164616481660 \\ & 1682170717091721172417481765177017901792181218141834183618531856185 \\ & 8190019021919194119441946196619681985201020122032203420542056209020 \\ & 932095211221152117213421562178224722492252225422882327 \end{aligned}$$

It may be noted that SU(3200) consists of 15016 digits.

(b) Other than the trivial 1, no other Ulam numbers have been found in the Smarandache U-sequence.

Open Problem:

(i) Can you find more primes in Smarandache U-sequence and are there infinitely many such primes?

(ii) Can you find more Ulam numbers in Smarandache U-sequence and are there infinitely many such Ulam numbers?

§4. Reversed Smarandache U-Sequence

It is defined as the sequence formed from the concatenation of Ulam numbers (U-sequence) written backward i.e. in reverse order. So, Reversed Smarandache U-sequence is

1, 21, 321, 4321, 64321, 864321, 11864321, 1311864321, 161311864321, 18161311864321, 2618161311864321, 282618161311864321,36282618161311864321, 3836282618161311864321, $\cdots$
Let us denote the nth term of the Reversed Smarandache U-sequence by RSU(n). So,

$$RSU(1) = 1$$

$$RSU(2) = 21$$

$$RSU(3) = 321$$

$$RSU(4) = 4321$$

$$RSU(5) = 64321 \text{ and so on.}$$

§5. Observations on reversed Smarandache U-sequence

(a) As against only 2 prime in Smarandache U-sequence, no primes in first 3200 terms of Reversed Smarandache U-sequence have been found.

(b) Other than the trivial 1, no Ulam number is known in the Reversed Smarandache U-sequence.

Open Problem:

(i) Can you find primes in Reversed Smarandache U-sequence and are there infinitely many such primes?

(ii) Can you find Ulam numbers in Reversed Smarandache U-sequence and are there infinitely many such Ulam numbers?

§6. Smarandache UP-Sequence

Smarandache sequence of Ulam Prime numbers or Smarandache UP-sequence is the sequence formed from concatenation of numbers in UP-sequence (Note that UP-sequence is the sequence of Ulam Prime numbers). So, Smarandache UP-sequence is
2, 23, 2311, 231113, 23111347, 2311134753, 231113475397, 231113475397131, 231113475397131197, 231113475397131197241, 231113475397131197241409, 231113475397131197241409431, 231113475397131197241409431607, 2311134753971311972414 09431607673, $\cdots$.

Let us denote the nth term of the Smarandache UP-sequence by SUP(n). So,

$$SUP(1) = 2$$

$$SUP(2) = 23$$

$$SUP(3) = 2311$$

$$SUP(4) = 231113$$

$$SUP(5) = 23111347 \textit{ and so on.}$$

§7. Observations on Smarandache UP-sequence

We have investigated Smarandache UP-sequence to find as to how many terms of Smarandache UP-sequence are primes?

We find that there are 5 primes in the first 245 terms of Smarandache UP-sequence. These are

SUP(1) = 2, SUP(2)=23, SUP(3)=2311, SUP(14) = 231113475397131197241409431607673 and

$$
\begin{aligned}
SUP(106) \quad = \quad & 2311134753971311972414094316076737397519839911103143314891531155317\\
& 0917212371239324472633278928332897304131093217337133733527354735933\\
& 6713691405741534211429743634409445145174519472949034969505950815531\\
& 6029648165696833691170437219729774597537755975837603769177278011810\\
& 18167853985738969897190139137931193779511961996439721974398519941102\\
& 4710271103691039110457104591050110567106571069110831109091108711197\\
& 1127311779.
\end{aligned}
$$

This prime consists of 413 digits.

It may be noted that SUP(245) consists of 1108 digits.

Can you find more primes in Smarandache UP-sequence and are there infinitely many such primes?

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A new Smarandache multiplicative function and its arithmetical properties

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Abstract In this paper, we introduce a new number theory function $D(n)$, then we use the elementary method to study the arithmetical properties of $D(n)$, and give several interesting conclusions for it.

Keywords Number theory function; Divisor function; Arithmetical properties.

§1. Introduction and results

For any positive integer n, the famous Dirichlet divisor function $d(n)$ is defined as the number of all distinct positive divisors of n. If $n = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdots p_r^{\alpha_r}$ be the prime power factorization of n, then from the definition and properties of $d(n)$ we may get

$$d(n) = (\alpha_1 + 1) \cdot (\alpha_2 + 1) \cdots (\alpha_r + 1).$$

From this formula we may immediately deduce that the first few values of $d(n)$ are $d(1) = 1$, $d(2) = 2$, $d(3) = 2$, $d(4) = 3$, $d(5) = 2$, $d(6) = 4$, $d(7) = 1$, $d(8) = 4 \cdots$. Now for any positive integer n, we use divisor function $d(n)$ to define a new number theory function $D(n)$ as follows: $D(n)$ denotes the smallest positive integer m such that n divide product $d(1)d(2) \cdots d(m)$. That is, $D(n) = \min \left\{ m : \ n | \prod\limits_{i=1}^{m} d(i) \right\}$. For example, $D(1) = 1$, $D(2) = 2$, $D(3) = 4$, $D(4) = 3$, $D(5) = 2^4$, $D(6) = 4$, $D(7) = 2^6$, $D(8) = 4$, $D(9) = 9$, $D(10) = 16$, $D(11) = 2^{10}$, $D(12) = 4$, $D(13) = 2^{12}$, $D(14) = 64$, $D(15) = 16$, $D(16) = 6$, $D(17) = 2^{16}$, $D(18) = 9$, $D(19) = 2^{18}$, $D(20) = 16 \cdots\cdots$. Recently, Professor Zhang Wenpeng asked us to study the arithmetical properties and the mean value properties of $D(n)$. About these problems, it seems that none had studied it yet, at least we have not seen any related papers before. I think these problems are interesting, because there are some close relations between $D(n)$ and the Dirichlet divisor function $d(n)$, so it can help us to find more information of $d(n)$. The main purpose of this paper is using the elementary method to study the calculating problem of $D(n)$, and give several interesting calculating formulae. That is, we shall prove the following:

Theorem 1. Let p be a prime, then we have the calculating formulae

A. $D(p) = 2^{p-1}$, for all prime p.

B. $D(3^2) = 3^2$, $D(p^2) = 2^{p-1} \cdot 3$, $D(3^3) = 2^2 \cdot 3$, $D(p^3) = 2^{p-1} \cdot 5$, if prime $p \geq 5$; $D(3^4) = 3^2 \cdot 2$, $D(5^4) = 3^4$, $D(p^4) = 2^{p-1} \cdot 7$, if $p \geq 7$; $D(3^5) = 2^2 \cdot 5$, $D(5^5) = 2^4 \cdot 7$,

$D(p^5) = 2^{p-1}\cdot 9$, $D(3^6) = 2^2\cdot 7$, $D(5^6) = 2^4\cdot 9$, $D(p^6) = 2^{p-1}\cdot 11$, if prime $p \geq 7$; $D(3^7) = 2^5$, $D(5^7) = 3^4\cdot 2$, $D(7^7) = 3^6$, $D(p^7) = 2^{p-1}\cdot 13$, if prime $p \geq 11$; $D(3^8) = 2^2\cdot 3^2$, $D(5^8) = 2^4\cdot 11$, $D(7^8) = 2^6\cdot 13$, $D(p^8) = 2^{p-1}\cdot 15$, if prime $p \geq 11$.

Theorem 2. If n be a square-free number (i.e., $p|n$ if and only if $p^2 \dagger n$), then $D(n) = 2^{P(n)-1}$, where $P(n)$ denotes the largest prime divisor of n.

Theorem 3. $D(n)$ is neither an additive function nor a multiplicative function, but $D(n)$ is a Smarandache multiplicative function.

§2. Proof of the theorems

In this section, we shall prove our Theorems directly. First we prove Theorem 1. We discuss the value distribution of $D(n)$ in the following two cases:
(i). If $n = p$ be a prime, let $D(p) = m$, then from the definition of $D(n)$ we have

$$p \mid \prod_{i=1}^{m} d(i), \quad p \nmid \prod_{i=1}^{j} d(i), \quad 0 < j < m.$$

So that $p \mid d(m)$. Let $m = p_1^{\alpha_1}\cdot p_2^{\alpha_2}\cdots p_r^{\alpha_r}$ be the prime power factorization of m, then $p \mid d(m) = (\alpha_1+1)(\alpha_2+1)\cdots(\alpha_r+1)$ implies that p divide one of $\alpha_i + 1$, where $1 \leq i \leq r$. So $m = 2^{p-1}$ be the smallest positive integer such that $p \mid d(m)$.

(ii). If $n = p^2$, where $p \geq 5$ be a prime, let $D(p^2) = m$. Then from the definition of $D(p^2)$ we can deduce that $p^2 \mid \prod_{i=1}^{m} d(i)$. So p^2 divide one of $d(1),\ d(2),\ \cdots,\ d(m)$, or p divide two of $d(1),\ d(2),\ \cdots,\ d(m)$. If p^2 divide one of $d(1),\ d(2),\ \cdots,\ d(m)$, then $m = 2^{p^2-1}$ or $m = 2^{p-1}\cdot 3^{p-1}$; If p divide two of $d(1),\ d(2),\ \cdots,\ d(m)$, then $m = 2^{p-1}\cdot 3$. So the smallest positive integer m such that $p^2 \mid d(1)d(2)\cdots d(m)$ is $m = 2^{p-1}\cdot 3$, since $2^{p-1}\cdot 3 < 3^{p-1} < 2^{2p-1}$.

If $p \geq 5$, then it is easy check the inequality

$$2^{p-1} < 3^{p-1} < 2^{2p-1} < 5^{p-1} < 7^{p-1} < 2^{3p-1} < 3^{2p-1},$$

so using the same method we can also obtain the calculating formula $D(3^2) = 3^2$, $D(p^2) = 2^{p-1}\cdot 3$, $D(3^3) = 2^2\cdot 3$, $D(p^3) = 2^{p-1}\cdot 5$, if prime $p \geq 5$; $D(3^4) = 3^2\cdot 2$, $D(5^4) = 3^4$, $D(p^4) = 2^{p-1}\cdot 7$, if $p \geq 7$; $D(3^5) = 2^2\cdot 5$, $D(5^5) = 2^4\cdot 7$, $D(p^5) = 2^{p-1}\cdot 9$, $D(3^6) = 2^2\cdot 7$, $D(5^6) = 2^4\cdot 9$, $D(p^6) = 2^{p-1}\cdot 11$, if prime $p \geq 7$; $D(3^7) = 2^5$, $D(5^7) = 3^4\cdot 2$, $D(7^7) = 3^6$, $D(p^7) = 2^{p-1}\cdot 13$, if prime $p \geq 11$; $D(3^8) = 2^2\cdot 3^2$, $D(5^8) = 2^4\cdot 11$, $D(7^8) = 2^6\cdot 13$, $D(p^8) = 2^{p-1}\cdot 15$, if prime $p \geq 11$.

This proves Theorem 1.

Now we prove Theorem 2. If n be a square-free number, let $n = p_1p_2\cdots p_k$, where $p_1 < p_2 < \cdots < p_k$ are k deferent primes. Suppose that $D(n) = m$, then it is clear that

$$n = p_1\cdot p_2\cdots p_k \mid \prod_{i=1}^{2^{p_k-1}} d(i) = d(1)\cdot d(2)\cdots d(2^{p_1-1})d(2^{p_2-1})\cdots d(2^{p_k-1}),$$

and n does not divide $d(1)d(2)d(3)\cdots d(m)$, if $m < 2^{p_k-1}$. So the smallest positive integer m such that $n = p_1\cdot p_2\cdots p_k$ divide $d(1)d(2)\cdots d(m)$ is $m = 2^{p_k-1}$. This proves Theorem 2.

Finally, we prove Theorem 3. It is clear that $D(1)+D(2)=3$, $D(1+2)=4$, $D(1)+D(2)\neq D(1+2)$; $D(1)+D(2)+D(3)=7$, $D(1+2+3)=4$, $D(1)+D(2)+D(3)\neq D(1+2+3)$. So $D(n)$ is not a additive function.

On the other hand, $D(1)\cdot D(2)\cdot D(3)=8$, $D(1\times 2\times 3)=D(6)=4$, $D(1)\cdot D(2)\cdot D(3)\neq D(1\times 2\times 3)$; $D(1)\cdot D(2)\cdot D(3)\cdot D(4)=8$, $D(1\times 2\times 3\times 4)=D(24)=5$, $D(1)\cdot D(2)\cdot D(3)\cdot D(4)\neq D(1\times 2\times 3\times 4)$. That is, $D(n)$ is not a multiplicative function. So $D(n)$ is neither an additive function nor a multiplicity function.

A number theory function $f(n)$ is called the Smarandache multiplicative function, if $f(n)=\max\{f(p_1^{\alpha_1}),\ f(p_2^{\alpha_2}),\ \cdots,\ f(p_k^{\alpha_k})\}$, where $n=p_1^{\alpha_1}\cdot p_2^{\alpha_2}\cdots p_k^{\alpha_k}$ be the prime power factorization of n. Now we prove that $D(n)$ be a Smarandache multiplicative function. In fact, let $m=\max\{D(p_1^{\alpha_1}),\ D(p_2^{\alpha_2}),\ \cdots,\ D(p_k^{\alpha_k})\}=D(p_s^{\alpha_s})$, then $D(p_i^{\alpha_i})\leq D(p_s^{\alpha_s})$ for all $i=1,\ 2,\ \cdots,\ k$. So $p_i^{\alpha_i}$ divide the product $d(1)d(2)\cdots d(m)$ for all $i=1,\ 2,\ \cdots,\ k$. Note that $(p_i^{\alpha_i},\ p_j^{\alpha_j})=1$, if $i\neq j$. So $n=p_1^{\alpha_1}\cdot p_2^{\alpha_2}\cdots p_k^{\alpha_k}$ also divide the product $d(1)d(2)\cdots d(m)$. That is means, $D(n)$ is a Smarandache multiplicative function. This completes the proof of Theorem 3.

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A predator-prey epidemic model with infected predator [1]

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Abstract A predator-prey epidemic model with infected predator is studied, and the effects of the disease on predation for the infected predator are not considered. A basic reproductive number which determines the outcome of the disease is given and the existence of the endemic equilibrium is discussed. By limiting system theory and Liapunov's stability, the necessary and sufficient conditions ensuring the global asymptotical stability of the disease-free equilibrium and the locally asymptotically stable of the endemic equilibrium is obtained.

Keywords Predator-prey, epidemic model, equilibrium, stability

§1. Introduction

The study which includes ecology and epidemiology is now termed as eco-epidemiology. Quite a number of studies have already been performed in eco-epidemilolgical systems [1, 2, 3]. It is practical and significant to study the combined model when one part of the recovered individuals can acquire permanent immunity while the other part has no immunity [4]. However, there are many disease with latent period [5, 6]. Thus, in this paper, we will consider a kind of the combination of $SEIR$ and $SEIS$ models in predator species. This paper is organized as follows: In section 2, the mathematical model is formulated. In section 3, the basic reproduction is obtained and the existence and stability of the endemic equilibrium is investigated.

§2. Model formulation

Let $X(t), Y(t)$ be the numbers of individuals in prey population and predator population at time t, respectively. The following is a predator-prey model:

$$\begin{cases} X' = X(a - bX - cY), \\ Y' = Y(hX - \mu). \end{cases} \tag{1}$$

where a, b, c, h, μ are positive constants.

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For system (1), the following lemma is well known [7].

Lemma 1. System (1) always has two equilibria $E_0(0,0)$ and $E_1(\frac{a}{b},0)$. Equilibrium E_0 is always unstable, equilibrium E_1 is unstable if $Q=\frac{b\mu}{ah}<1$ and is globally asymptotically stable(GAS) if $Q=\frac{b\mu}{ah}>1$; When $Q=\frac{b\mu}{ah}<1$, system (1) also has a unique positive equilibrium $E_2(\frac{\mu}{h},a(1-Q)/c)$ which is GAS.

Assume that the disease only spreads among the predator species, and that the effects of the disease on predation for the infected predator are not considered. The total population of predator $Y(t)$ is divided into four subpopulation: the susceptible $S(t)$, the exposed $E(t)$, the infectious $I(t)$, and the recovered $R(t)$. $Y(t)=S(t)+E(t)+I(t)+R(t)$. Assume that the disease has the vertical transmission for the exposed and the infectious individuals, and that the corresponding proportions are $q,p(0<q,p<1)$ respectively. So we will consider the combination of $SEIR$ and $SEIS$ model with bilinear infection rate βSI as follows:

$$\begin{cases} X'=X(a-bX)-cXS-cXI-cXE-cXR,\\ S'=hXS+(1-p)hXI+(1-q)hXE+hXR-\beta SI-\mu S+\delta I,\\ E'=phXI+qhXE+\beta SI-\epsilon E-\mu E,\\ I'=\epsilon E-\gamma I-\mu I-\delta I,\\ R'=\gamma I-\mu R. \end{cases} \tag{2}$$

Here ϵ is the transfer rate constant from the exposed subpopulation to the infectious subpopulation, δ is the transfer rate constant from the infective subpopulation to the susceptible subpopulation, and γ is the transfer rate constant from the infective subpopulation to the recovered subpopulation.

Substituting $R=Y-S-E-I$ into the first two equations in (2) gives the following equations

$$\begin{cases} X'=X(a-bX)-cXY,\\ Y'=Y(hX-\mu),\\ S'=hXS+(1-p)hXI+(1-q)hXE+hX(Y-S-E-I)-\beta SI-\mu S+\delta I,\\ E'=phXI+qhXE+\beta SI-\epsilon E-\mu E,\\ I'=\epsilon E-\gamma I-\mu I-\delta I. \end{cases} \tag{3}$$

Obviously, it follows from Lemma 1 that, when $Q>1$, $\lim\limits_{t\to\infty}S(t)=\lim\limits_{t\to\infty}E(t)=\lim\limits_{t\to\infty}I(t)=\lim\limits_{t\to\infty}R(t)=0$ and $\lim\limits_{t\to\infty}X(t)=\frac{a}{b}$. Further, when $Q<1$, the system consisting of the last three equations in (3) has the following limiting system

$$\begin{cases} S'=\mu K-\beta SI-p\mu I-q\mu E-\mu S+\delta I,\\ E'=p\mu I+q\mu E+\beta SI-\epsilon E-\mu E,\\ I'=\epsilon E-\gamma I-\mu I-\delta I, \end{cases} \tag{4}$$

where $K=\frac{a(1-Q)}{c}$.

In the following, we consider the dynamical behavior of (4), which is equivalent with that of (3).

§3. The existence and the stability of equilibria

Denote $N = S + E + I$, it follows from (4) that $N' = \mu K - \mu N - \gamma I \leq \mu(K - N)$, so $\limsup_{t\to\infty} N(t) \leq K$. Therefore, the set $\Omega = \{(S, E, I) \in R_+^3 | S+E+I \leq K\}$ is positively invariant to (4).

By straightforward calculation, there are the following result on the existence of equilibrium for (4).

Theorem 1. Denote $R_0 = \dfrac{\beta\epsilon K}{(\mu+\gamma+\delta)(\mu+\epsilon-q\mu)-p\mu\epsilon}$. System (4) always exists the disease-free equilibrium $E_2^o(K, 0, 0) \in \Omega$. And system also has a unique endemic equilibrium $E_2^*(S^*, E^*, I^*) \in \Omega$ if $R_0 > 1$, where

$$S^* = \frac{\mu K - \dfrac{(\mu+\epsilon)(\mu+\gamma+\delta)I^*}{\epsilon}}{\mu}, E^* = \frac{\mu+\gamma+\delta}{\varepsilon}I^*,$$
$$I^* = \frac{\mu((\mu+\gamma+\delta)(\mu+\epsilon-q\mu)-p\epsilon\mu)}{\beta(\mu+\gamma+)(\mu+\gamma)}(R_0 - 1).$$

On the stability of equilibria of (4), we have

Theorem 2. The disease-free equilibrium E_2^o is GAS in Ω if $R_0 < 1$ and it is unstable if $R_0 > 1$. The endemic equilibrium E_2^* is locally asymptotically stable(LAS) if it exists.

Proof. Let $V = \epsilon E + (\mu + \epsilon - q\mu)I$, then straightforward calculation shows that $V' < 0$ if $R_0 < 1$. Therefore, It follows from Liapunov stability theorem that the equilibrium E_2^o is GAS if $R_0 < 1$.

Jacobian matrix of (4) at E_2^o is

$$J(E_2^o) = \begin{pmatrix} -\mu & -q\mu & -\beta K - p\mu - \delta \\ \beta I & q\mu - \mu - \epsilon & \beta K + p\mu \\ 0 & \epsilon & -(\mu+\gamma+\delta) \end{pmatrix},$$

and $\det(\lambda I - J(E_2^o)) = (\lambda+\mu)[\lambda^2+\lambda(2\mu+\epsilon+\gamma-q\mu+\delta)+(\mu+\gamma+\delta)(\mu+\epsilon-q\mu)-\epsilon(p\mu+\beta K)] = 0$. Since $(2\mu + \epsilon + \gamma - q\mu + \delta) > 0$ for $0 < q < 1$, then all eigenvalues have negative real parts if and only if $(\mu + \gamma + \delta)(\mu + \epsilon - q\mu) - \epsilon(p\mu + \beta K) > 0$, that is $R_0 < 1$. Therefore E_2^o is unstable if $R_0 > 1$.

Jacobian matrix of (4) at E_2^* is

$$J(E_2^*) = \begin{pmatrix} -\beta I^* - \mu & -q\mu & \beta S^* - p\mu - \delta \\ \beta I^* & q\mu - (\mu+\epsilon) & \beta S^* + p\mu \\ 0 & \epsilon & -(\mu+\gamma+\delta) \end{pmatrix},$$

and $\det(\lambda I - J(E_2^*)) = \lambda^3 + a_1\lambda^2 + a_2\lambda + a_3$, where $a_1 = \beta I^* + (3 - q)\mu + \epsilon + \gamma + \delta > 0$, $a_2 = (\mu+\gamma+\delta)(\mu+\beta I^*)+\mu(\mu+\epsilon-q\mu)+\beta I^*(\mu+\epsilon) > 0, a_3 = \beta I^*(\mu+\epsilon)(\mu+\gamma+\delta)+\beta I^*\epsilon\delta >$

0. Straightforward calculation shows that $a_1a_2 - a_3 > 0$. It follows from Routh-Hurwitz Criterion[8] that the endemic equilibrium E_2^* is LAS when $R_0 > 1$.

This completes the proof.

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An equation involving the Lucas numbers and Smarandache primitive function

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Abstract For any positive integer n, let $S_p(n)$ denotes the Smarandache primitive function, L_n denotes the Lucas numbers. The main purpose of this paper is using the elementary methods to study the number of the solutions of the equation $S_p(L_1)+S_p(L_2)+\cdots+S_p(L_n) = S_p(L_{n+2}-3)$, and give all positive integer solutions for this equation.

Keywords Lucas numbers, Smarandache primitive function, equation, solutions.

§1. Introduction

As usual, the Lucas sequence L_n is defined by the second-order linear recurrence sequence $L_{n+2} = L_{n+1}+L_n$ for $n \geq 1, L_1 = 1, L_2 = 3$. This sequence plays a very importment role in the studied of theory and application of mathmatics. Therefore, the various properties of L_n was investigated by many authors. For example, J.L.Brown [1] studied the unique representation of intergers as sums of Distinct Lucas Numbers. Wenpeng Zhang [2] obtained some identities involving the Lucas numbers.

Let p be a prime, n be any positive integer. The Smarandache primitive function $S_p(n)$ is defined as the smallest positive integer such that $S_p(n)!$ is divisible by p^n. For example, $S_2(1) = 2, S_2(2) = S_2(3) = 4, S_2(4) = 6, \cdots$. In problem 47, 48 and 49 of book [3], the famous Rumanian-born American number theorist, Professor F.Smarandache asked us to study the properties of the $S_p(n)$. There are closely relations between the Smarandache primitive function $S_p(n)$ and the famous function $S(n)$, where

$$S(n) = \min\{m : m \in N, n \mid m!\}.$$

From the definition of $S(n)$, obviously we have $S(p) = p$, and if $n \neq 4, n \neq p$, then $S(n) < n$. So we have

$$\pi(x) = -1 + \sum_{i=2}^{[x]} \left[\frac{S(n)}{n}\right],$$

where $\pi(x)$denotes the number of primes which less than x.

The research on Smarandache function $S(n)$, Smarandache primitive function $S_p(n)$ and the equations involving Smarandache primitive function $S_p(n)$ is an significant and important problem in Number Theory. Therefore, many scholars and researchers have studied them before,

see reference [4-6]. Professor Zhang [7] have obtained an interesting asymptotic formula. That is, for any fixed prime p and any positive integer n, we have

$$S_p(n) = (p-1)n + O\left(\frac{p}{\ln p} \cdot \ln n\right).$$

Li Jie [8] studied the solvability of the equation $S_p(1) + S_p(2) + \cdots + S_p(n) = S_p\left(\frac{n(n+1)}{2}\right)$, and gave its all positive integer solutions. But it seems that no one knows the relationship between the Lucas numbers and the Smarandache primitive function. In this paper, we use the elementary methods to study the solvability of the equation

$$S_p(L_1) + S_p(L_2) + \cdots + S_p(L_n) = S_p(L_{n+2} - 3),$$

and give all positive integer solutions for this equation. That is, we will prove the following:

Theorem. Let p be a given prime, n be any positive integer, then the equation

$$S_p(L_1) + S_p(L_2) + \cdots + S_p(L_n) = S_p(L_{n+2} - 3) \tag{1}$$

has finite solutions. They are $n = 1, 2, \cdots, n_p$, where

$$n_p = \left[\frac{\log\left((p+3) + \sqrt{(p+3)^2+4}\right) - \log 2}{\log(1+\sqrt{5}) - \log 2} - 2\right],$$

$[x]$ denotes the biggest integer $\leq x$.

Especially, taking $p = 3, 5, 11$, we may immediately duduce the following:

Corollary 1. All of the positive integer solutions for the equation

$$S_3(L_1) + S_3(L_2) + \cdots + S_3(L_n) = S_3(L_{n+2} - 3)$$

are $n = 1$.

Corollary 2. All of the positive integer solutions for the equation

$$S_5(L_1) + S_5(L_2) + \cdots + S_5(L_n) = S_5(L_{n+2} - 3)$$

are $n = 1, 2$.

Corollary 3. All of the positive integer solutions for the equation

$$S_{11}(L_1) + S_{11}(L_2) + \cdots + S_{11}(L_n) = S_{11}(L_{n+2} - 3)$$

are $n = 1, 2, 3$.

§2. Several lemmas

To complete the proof of the theorem, we need the following several simple lemmas.

Lemma 1. Let p be a prime, n be any positive integer, $S_p(n)$denote the Smarandache primitive function, then we have

$$S_p(k) \begin{cases} = pk, & \text{if} \quad k \leq p, \\ < pk, & \text{if} \quad k > p. \end{cases}$$

Proof. (See reference [9]).

Lemma 2. Let L_n be the Lucas sequence with $L_1 = 1$ and $L_2 = 3$, then we have the identity

$$L_1 + L_2 + \cdots + L_n = L_{n+2} - 3.$$

Proof. From the second-order linear recurrence sequence $L_{n+2} = L_{n+1} + L_n$ we can easilly get the identity of Lemma 2.

Lemma 3. Let p be a prime, n be any positive integer, if n and p satisfing $p^\alpha \parallel n!$, then

$$\alpha = \sum_{i=1}^{\infty} \left[\frac{n}{p^i}\right].$$

Proof. (See reference [10]).

Lemma 4. Let p be a prime, n be any positive integer. Then there must exist a positive integer M_k with $1 \leq M_k \leq L_k (k = 1, 2, \cdots, n)$ such that

$$S_p(L_1) = M_1 p, \quad S_p(L_2) = M_2 p, \quad \cdots, \quad S_p(L_n) = M_n p.$$

and

$$L_k \leq \sum_{i=1}^{\infty} \left[\frac{M_k p}{p^i}\right].$$

Proof. From the definition of $S_p(n)$, Lemma 1 and Lemma 3, we can easilly get the conclusions of Lemma 4.

§3. Proof of the theorem

In this section, we will complete the proof of Theorem.

First, if $p = 2$, then the equation (1) is $S_2(L_1) + S_2(L_2) + \cdots + S_2(L_n) = S_2(L_{n+2} - 3)$.

(i)If $n = 1$, $S_2(L_1) = 2 = S_2(L_3 - 3)$, so $n = 1$ is the solution of the equation (1).

(ii)If $n = 2$, $S_2(L_1) + S_2(L_2) = 2 + 2 \times 2 = S_2(L_4 - 3)$, so $n = 2$ is the solution of the equation (1).

(iii) If $n = 3$, $S_2(L_1) + S_2(L_2) + S_2(L_3) = 2 + 2 \times 2 + 3 \times 2 = 12$, but $S_2(L_5 - 3) = S_2(8) = 10$, so $n = 3$ is not the solution of the equation (1).

(iv) If $n > 3$, then from Lemma 3 we know that there must exist a positive integer M_k with $1 \leq M_k \leq L_k (k = 1, 2, \cdots, n)$ such that

$$S_2(L_1) = 2M_1, S_2(L_2) = 2M_2, \cdots, S_2(L_n) = 2M_n.$$

So we have $S_2(L_1) + S_2(L_2) + \cdots + S_2(L_n) = 2(M_1 + M_2 + \cdots + M_n)$.

On the other hand, notice that $M_1 = 1$, $M_2 = 2$, $M_3 = 3$, then from Lemma 3 we have

$$\begin{aligned}
& \sum_{i=1}^{\infty} \left[\frac{2(M_1 + M_2 + \cdots + M_n) - 1}{2^i}\right] \\
= & \sum_{i=1}^{\infty} \left[\frac{2(M_1 + M_2 + \cdots + M_n - 1) + 1}{2^i}\right] \\
= & M_1 + M_2 + \cdots + M_n - 1 + \sum_{i=2}^{\infty} \left[\frac{2(M_1 + M_2 + \cdots + M_n - 1) + 1}{2^i}\right] \\
\geq & M_1 + M_2 + \cdots + M_n - 1 + \sum_{i=2}^{\infty} \left[\frac{2(M_1 + M_2 + M_3 - 1) + 1}{2^i}\right] \\
+ & \sum_{i=1}^{\infty} \left[\frac{M_4 + M_5 + \cdots + M_n}{2^i}\right] \\
\geq & M_1 + (M_2 + 1) + (M_3 + 1) + \left(M_4 + \sum_{i=1}^{\infty} \left[\frac{M_4}{2^i}\right]\right) \\
& + \cdots + \left(M_n + \sum_{i=1}^{\infty} \left[\frac{M_n}{2^i}\right]\right) \\
\geq & L_1 + L_2 + L_3 + \sum_{i=1}^{\infty} \left[\frac{2M_4}{2^i}\right] + \cdots + \sum_{i=1}^{\infty} \left[\frac{2M_n}{2^i}\right] \\
\geq & L_1 + L_2 + L_3 + L_4 + \cdots + L_n \\
= & L_{n+2} - 3
\end{aligned}$$

then from Lemma 2 we can get

$$2^{L_{n+2}-3} \mid (2(M_1 + M_2 + \cdots + M_n) - 1)!$$

Therefore,

$$\begin{aligned}
S_2(L_{n+2} - 3) & \leq 2(M_1 + M_2 + \cdots + M_n) - 1 \\
& < 2(M_1 + M_2 + \cdots + M_n) \\
& = S_2(L_1) + S_2(L_2) + \cdots + S_2(L_n).
\end{aligned}$$

so there is no solutions for the equation (1) in this case.

If $p \geq 3$ we will discuss the solutions of the equation $S_p(L_1) + S_p(L_2) + \cdots + S_p(L_n) = S_p(L_{n+2} - 3)$ in the following three cases:

(i) If $L_{n+2} - 3 \leq p$, solving this inequality we can get $1 \leq n \leq n_p$, where

$$n_p = \left[\frac{\log\left((p+3) + \sqrt{p^2 + 6p + 10} - \log 2\right)}{\log(1 + \sqrt{5}) - \log 2} - 2\right], \tag{2}$$

$[x]$ denotes the biggest integer $\leq x$.

Then from Lemma 1 we have

$$S_p(L_{n+2} - 3) = p(L_{n+2} - 3).$$

Noting that $1 \leq n \leq n_p$, that is $L_n \leq L_{n+2} - 1 \leq p$, from Lemma 1 and Lemma 2 we can get

$$S_p(L_1) + S_p(L_2) + \cdots + S_p(L_n) = pL_1 + pL_2 + \cdots + pL_n = p(L_{n+2} - 3)$$

Combining above two formulae, we may easily get $n = 1, 2, \cdots, n_p$ are the solutions of the equation $S_p(L_1) + S_p(L_2) + \cdots + S_p(L_n) = S_p(L_{n+2} - 3)$.

(ii) If $L_n \leq p < L_{n+2} - 3$, solving this inequality we can get $n_p < n \leq N_p$, where

$$N_p = \left[\frac{\log\left(\sqrt{5}p + \sqrt{5p^2 + 4}\right) - \log 2}{\log(1 + \sqrt{5}) - \log 2} \right], \tag{3}$$

$[x]$ denotes the biggest integer $\leq x$.

Then from Lemma 1 we have

$$S_p(L_{n+2} - 3) < p(L_{n+2} - 3).$$

But

$$S_p(L_1) + S_p(L_2) + \cdots + S_p(L_n) = pL_1 + pL_2 + \cdots + pL_n = p(L_{n+2} - 3).$$

Hence the equation (1) has no solution in this case.

(iii) If $n \geq N_p + 1$, that is $p < L_n < L_{n+2} - 1$.

Now from Lemma 4 there must exist a positive integer M_k with $1 \leq M_k \leq L_k (k = 1, 2, \cdots, n)$ such that

$$S_p(L_1) = M_1 p, \quad S_p(L_2) = M_2 p, \quad \cdots, \quad S_p(L_n) = M_n p.$$

Then we have

$$S_p(L_1) + S_p(L_2) + \cdots + S_p(L_n) = p(M_1 + M_2 + \cdots + M_n). \tag{4}$$

(a) If $L_{N_p+1} = p + 1$, then $L_{N_p+2} - 4 = L_{N_p+1} + L_{N_p} - 4 = L_{N_p} + p - 3 \geq p$.

Notice that $M_1 = L_1, M_2 = L_2, \cdots, M_{N_p} = L_{N_p}$, from Lemma 4 we also have

$$
\begin{aligned}
& \sum_{i=1}^{\infty}\left[\frac{(M_1+M_2+\cdots+M_n)p-1}{p^i}\right] \\
= & \sum_{i=1}^{\infty}\left[\frac{p(M_1+M_2+\cdots+M_n-1)+p-1}{p^i}\right] \\
= & M_1+M_2+\cdots+M_n-1+\sum_{i=2}^{\infty}\left[\frac{p(M_1+M_2+\cdots+M_n-1)+p-1}{p^i}\right] \\
\geq & M_1+M_2+\cdots+M_n-1+\sum_{i=2}^{\infty}\left[\frac{p(M_1+M_2+\cdots+M_{N_p}-1)+p-1}{p^i}\right] \\
& +\sum_{i=2}^{\infty}\left[\frac{p(M_{N_p+1}+M_{N_p+2}+\cdots+M_n)}{p^i}\right] \\
= & M_1+M_2+\cdots+M_n-1+\sum_{i=2}^{\infty}\left[\frac{p(L_{N_p+2}-4)+p-1}{p^i}\right]+\sum_{i=1}^{\infty}\left[\frac{M_{N_p+1}+\cdots+M_n}{p^i}\right] \\
\geq & M_1+M_2+\cdots+M_{N_p}+\left(M_{N_p+1}+\sum_{i=1}^{\infty}\left[\frac{M_{N_p+1}}{p^i}\right]\right)++\cdots+\left(M_n+\sum_{i=1}^{\infty}\left[\frac{M_n}{p^i}\right]\right) \\
\geq & L_1+L_2+\cdots+L_{N_p}+\sum_{i=1}^{\infty}\left[\frac{M_{N_p+1}p}{p^i}\right]+\cdots+\sum_{i=1}^{\infty}\left[\frac{M_np}{p^i}\right] \\
\geq & L_1+L_2+\cdots+L_n \\
= & L_{n+2}-3.
\end{aligned}
$$

Then from Lemma 3 we can get

$$p^{L_{n+2}-3} \mid (p(M_1+M_2+\cdots+M_n)-1)!.$$

Therefore,

$$
\begin{aligned}
S_p(L_{n+2}-3) & \leq p(M_1+M_2+\cdots+M_n)-1 \\
& < p(M_1+M_2+\cdots+M_n) \\
& = S_p(L_1)+S_p(L_2)+\cdots+S_p(L_n).
\end{aligned}
$$

Hence the equation (1) has no solution in this case.

(b) If $L_{N_p+1} > p+1$, then $p < M_{N_p+1} \leq L_{N_p+1} = L_{N_p} + L_{N_p-1} < 2p$.

Notice that $M_1 = L_1, M_2 = L_2, \cdots, M_{N_p} = L_{N_p}$ and $p < M_{N_p+1} < 2p$ from Lemma 4 we

also have

$$\begin{aligned}
& \sum_{i=1}^{\infty}\left[\frac{(M_1+M_2+\cdots+M_n)p-1}{p^i}\right] \\
= & \sum_{i=1}^{\infty}\left[\frac{p(M_1+M_2+\cdots+M_n-1)+p-1}{p^i}\right] \\
= & M_1+M_2+\cdots+M_n-1+\sum_{i=2}^{\infty}\left[\frac{p(M_1+M_2+\cdots+M_n-1)+p-1}{p^i}\right] \\
\geq & M_1+M_2+\cdots+M_n-1+\sum_{i=2}^{\infty}\left[\frac{p(M_1+M_2+\cdots+M_{N_p})+p-1}{p^i}\right] \\
& +\sum_{i=2}^{\infty}\left[\frac{p(M_{N_p+1}+M_{N_p+2}+\cdots+M_n-1)}{p^i}\right] \\
= & M_1+M_2+\cdots+M_n-1+\sum_{i=2}^{\infty}\left[\frac{p(L_{N_p+2}-3)+p-1}{p^i}\right]+\sum_{i=1}^{\infty}\left[\frac{M_{N_p+1}+\cdots+M_n-1}{p^i}\right] \\
\geq & M_1+M_2+\cdots+M_{N_p}+\left(M_{N_p+1}+\sum_{i=1}^{\infty}\left[\frac{M_{N_p+1}-1}{p^i}\right]\right)+\cdots+\left(M_n+\sum_{i=1}^{\infty}\left[\frac{M_n}{p^i}\right]\right) \\
\geq & L_1+L_2+\cdots+L_{N_p}+\sum_{i=1}^{\infty}\left[\frac{M_{N_p+1}p}{p^i}\right]+\cdots+\sum_{i=1}^{\infty}\left[\frac{M_np}{p^i}\right] \\
\geq & L_1+L_2+\cdots+L_n \\
= & L_{n+2}-3.
\end{aligned}$$

Then from Lemma 3 we can get

$$p^{L_{n+2}-3} \mid (p(M_1+M_2+\cdots+M_n)-1)!.$$

Therefore,

$$\begin{aligned}
S_p(L_{n+2}-3) & \leq p(M_1+M_2+\cdots+M_n)-1 \\
& < p(M_1+M_2+\cdots+M_n) \\
& = S_p(L_1)+S_p(L_2)+\cdots+S_p(L_n).
\end{aligned}$$

Hence $n \geq N_p+1$ the equation (1) has no solution.

Now the Theorem follows from (i), (ii), and (iii).

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Two rings in Is-algebras

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Abstract Let X be a IS-algebra(BCI-Semigroup), $\mathrm{R(X)} = \{x \in \mathrm{X} | 0 * x = x\}$ be called ring part of X, $AR(\mathrm{X}) = \{x \in \mathrm{X} | 0 * (0 * x) = x\}$ be called adjoint ring part of X. They are subalgebras and rings in X, and some equivalent conditions of $R(\mathrm{X})$ and $AR(\mathrm{X})$ being ideals are shown.

Keywords IS-algebra ideal, subalgebra, ring part.

§1. Basic concepts

Mathematician K.Iseki of Japan introduced the BCI–algebra in1966，Mathematician Y.B.Jun in Korea introduced the BCI-semigroup(IS-algebra) in 1993. We show some relational definitions and conclusions for convenience of discussion.

Definition 1[1]. An algebra $(\mathrm{X}, *, 0)$ of $(2, 0)$ is said to be a BCI-algebra if it satisfies:
(1) $((x * y) * (x * z)) * (z * y) = 0$.
(2) $(x * (x * y)) * y = 0$.
(3) $x * x = 0$.
(4) $x * y = 0$ and $y * x = 0$ imply $x = y$.

In BCI–algebra，we have the following basic formulas:
(1) $(x * y) * z = (x * z) * y$.
(2) $x * 0 = x$.
(3) $0 * (x * y) = (0 * x) * (0 * y)$.
(4) $0 * (0 * (0 * x)) = 0 * x$.

Definition 2[2]. A IS-algebra $(\mathrm{X}, *, \cdot, 0)$ is a non-empty set X with two operation “$*$” and “$\cdot$”, and with a constant element 0 such that following axioms are satisfied: for all x, y, $z \in \mathrm{X}$, we have
(1)$(\mathrm{X}, *, 0)$ is BCI-algebra.
(2) $(\mathrm{X}, \cdot)$ is semigroup.
(3) Distributive law: $x \cdot (y * z) = (x \cdot y) * (x \cdot z)$,$(x * y) \cdot z = (x \cdot z) * (y \cdot z)$.

$x \cdot y$ is usual to be written xy and IS-algebra $(\mathrm{X}, *, \cdot, 0)$ is usual to be written X for short.

In IS-algebra X, we have $0x = x0 = 0$[2].

Let Y be the non-empty subset of IS-algebra X, if “$*$” and “$\cdot$” are closed in Y, then $(\mathrm{Y}, *, \cdot, 0)$ is IS-algebra too, we call it is a subalgebra of X[2].

Definition 3. [3]. I is non-empty subset of IS-algebra X, It is said to be ideal of X, if
(1) $\forall x \in$ X, $\forall a \in$ I, we have $xa, ax \in$ I.
(2) $\forall x, y \in$ X, if $x * y \in$ I and $y \in$ I, then $x \in$ I.
It is easy to prove that $(\mathrm{R}, -, \cdot, 0)$ is IS-algebra in Ring $(\mathrm{R}, +, \cdot)$, the ideal of R as Ring is agree with the ideal of R as IS-algebra.

§2. Ring part of Is-algebra

Similar with BCI-G part of BCI-algebra, we introduce some relational definition.

Definition 4. Let $(\mathrm{X}, *, \cdot, 0)$ be IS-algebra, $R(\mathrm{X}) = \{x \in \mathrm{X} | 0 * x = x\}$ is said to be ring part of X.

Clearly, Y is ring part of IS-algebra $(\mathrm{X}, *, \cdot, 0)$ if and only if Y is G part of BCI-algebra X. we call $R(\mathrm{X})$ is ring part of X, because the following conclusion:

Theorem 1. Ring part $R(\mathrm{X})$ of IS-algebra X is subalgebra of X and ring that character is 2. Also, $R(\mathrm{X})$ is maximal ring about operation "$*$" and "$\cdot$".

Proof. $\forall x, y \in R(\mathrm{X})$, We have
$0 * (x * y) = (0 * x) * (0 * y) = x * y, 0 * (xy) = (0y) * (xy) = (0 * x)y = xy$, So $x * y$, $xy \in R(\mathrm{X})$, in other words, $R(\mathrm{X})$ is subalgebra of X.
Since $R(\mathrm{X})$ is combination part of BCI-algebra, then $(R(\mathrm{X}), *, 0)$ certainly be combination BCI-algebra[6], so it is Abel group that 0 is identity element. Also, since multiplication is closed in $R(\mathrm{X})$ and $x * x = 0$, then $(R(\mathrm{X}), *, \cdot)$ is a ring which character is 2.

Let Y be ring in X about operation "$*$" and "$\cdot$", $\forall y \in \mathrm{Y} \subseteq \mathrm{X}$, because $0 * y = y * 0 = y$, Then $y \in R(\mathrm{X})$, and finally $\mathrm{Y} \subseteq R(\mathrm{X})$.

But generally speaking, $R(\mathrm{X})$ may not be ideal of X.

Example 1. Let X $= 0, a, b$, put $x \cdot y = 0$, and operation "$*$" is following:

$*$	0	a	b
0	0	0	b
a	0	0	b
b	b	b	0

Then $(\mathrm{X}, *, \cdot)$ is IS-algebra and $R(\mathrm{X}) = \{0, b\}$. However, $R(\mathrm{X})$ is not the ideal of X.

Following, we discuss the condition of $R(\mathrm{X})$ to be ideal.

Lemma 1. $R(\mathrm{X})$ is ideal of IS-algebra $(\mathrm{X}, *, \cdot, 0)$ if and only if $R(\mathrm{X})$ is ideal of BCI-algebra $(\mathrm{X}, *, 0)$.

Proof. The necessity is clearly, we discuss sufficient part as follow.

Let $R(\mathrm{X})$ be ideal of BCI-algebra $(\mathrm{X}, *, \cdot)$, for every $x \in \mathrm{X}$, $a \in R(\mathrm{X})$, we have

$$0 * (ax) = (0x) * (ax) = (0 * a)x = ax,$$

Therefore $ax \in R(\mathrm{X})$, by the same reasoning $xa \in R(\mathrm{X})$, thus, $R(\mathrm{X})$ is ideal of BCI-algebra. Put $(\mathrm{X}, *, 0)$ is BCI-algebra, a is a element in X, in reference [5], it gives us the

following mapping:

$$a_r : x \to x * a(\forall x \in \mathrm{X}),$$

and the following conclusion:

Lemma 2.[5] Let $G(\mathrm{X})$ be G-part (connection part) of BCI-algebra X, The following conditions are equivalent:
(1) $G(\mathrm{X})$ is ideal of X.
(2) $\forall a \in \mathrm{X}, a_r : x \to x * a$ is injective.
(3) $\forall a, b \in \mathrm{X},\ a_r b_r = (a * b)_r = (b * a)_r$.
(4) $\forall x, y \in \mathrm{X},\ \forall a \in R(\mathrm{X})$, if $x * a = y * a$, then $x = y$.

We obtain the following conclusion by lemma 1 and lemma 2.

Theorem 2. Let $R(\mathrm{X})$ be ring part of IS-algebra $(\mathrm{X}, *, \cdot, 0)$, then the following conditions are equivalent:
(1) $R(\mathrm{X})$ is ideal of IS-algebra $(\mathrm{X}, *, \cdot, 0)$.
(2) $\forall a \in \mathrm{X},\ a_r : x \to x * a$ is injective.
(3) $\forall a, b \in \mathrm{X},\ a_r b_r = (a * b)_r = (b * a)_r$.
(4) $\forall x, y \in \mathrm{X},\ \forall a \in R(\mathrm{X})$, if $x * a = y * a$, then $x = y$.

§3. Adjoint ring part of IS-algebra

Similar with p-semi simple part[4] (generalized connection part[6]), we take relational definitions to IS-algebra.

Definition 5. In IS-algebra $(\mathrm{X}, *, \cdot, 0)$. it is said $\mathrm{AR(X)} = \{x \in \mathrm{X} | 0 * (0 * x) = x\}$ to be adjoint ring part of X.

Example 2. Let $\mathrm{X} = \{0, a, b, c\}$, $xy = 0$, and operation "$*$" is following:

$*$	0	a	b	c
0	0	c	0	a
a	a	0	a	c
b	b	c	0	a
c	c	0	c	a

Then $(\mathrm{X}, *, \cdot, 0)$ is IS-algebra and $R(\mathrm{X}) = \{0\}$, $AR(\mathrm{X}) = \{0, a, c\}$. Clearly Y is adjoint ring part of IS algebra $(\mathrm{X}, *, \cdot, 0)$ if and only if Y is p-half singe part of BCI-algebra $(\mathrm{X}, *, 0)$. In IS-algebra X, put $x + y = x * (0 * y)$, $\forall x, y \in AR(\mathrm{X})$, we have

$$0 * (0 * (x + y)) = 0 * (0 * (x * (0 * y))) = (0 * (0 * x)) * (0 * (0 * (0 * y))) = x * (0 * y) = x + y.$$

Therefore, $x + y \in AR(\mathrm{X})$, then this addition can be operation of $AR(\mathrm{X})$.

Theorem 3. [6]In generalized connection BCI-algebra $(\mathrm{X}, *, 0)$, let $x + y = x * (0 * y)$, then $(\mathrm{X}, +)$ is Abel group that 0 is the zero element.

Theorem 4. In IS-algebra $(\mathrm{X}, *, 0)$, we have
(1) $AR(\mathrm{X})$ is subalgebra of $(\mathrm{X}, *, 0)$
(2) $AR(\mathrm{X})$ is maximal ring of X about "+" and "·" .

Proof. (1) $\forall x, y \in AR(\mathrm{X})$, we have

$$0 * (0 * (x * y)) = (0 * (0 * x)) * (0 * (0 * y)) = x * y,$$

$$0 * (0 * (xy)) = (0y) * (0y) * (xy)) = (0 * (0 * x))y = xy,$$

That is $x * y$, $xy \in AR(\mathrm{X})$, so $AR(\mathrm{X})$ is subalgebra of $(\mathrm{X}, *, \cdot, 0)$.

(2) $(AR(\mathrm{X}), *, 0)$ is BCI-algebra by (1), and is generalized connection. For lemma 3, $(AR(\mathrm{X}), +)$ is Abel group that 0 is zero element. Also

$$(x + y)z = (x * (0 * y))z = (xz) * ((0z) * (yz)) = (xz) * (0 * (yz)) = xz + yz$$

For same reasoning, $x(y + z) = xy + xz$, thus $(AR(\mathrm{X}), +, \cdot)$ is ring. Let $(\mathrm{Y}, +, \cdot)$ be ring in X, $\forall y \in \mathrm{Y}$, $y + 0 = y * (0 * 0) = y$, hence $0 + y = y$, that is $0 * (0 * y) = y$, therefore $y \in AR(\mathrm{X})$, finally $\mathrm{Y} \subseteq AR(\mathrm{X})$. Next, we discuss the condition that $R(\mathrm{X})$ to be ideal.
Similar with lemma 1, we have the following conclusion:

Lemma 4. $AR(\mathrm{X})$ is ideal of IS-algebra $(\mathrm{X}, *, \cdot, 0)$ if and only if $AR(\mathrm{X})$ is ideal of BCI-algebra $(\mathrm{X}, *, 0)$.

Lemma 5.[4] The following conditions are equivalent in BCI-algebra $(\mathrm{X}, *, 0)$:
(1) p-half single part $SP(\mathrm{X})$ is ideal of X.
(2) $\forall a \in SP(\mathrm{X})$, $a_r : x \to x * a$ is injective.
(3) $\forall \upsilon \in \mathrm{X}$, $\nu \in SP(\mathrm{X})$, have $(\upsilon * \nu) * (0 * \nu) = \upsilon$.
We can obtain the following conclusion by lemma 4 and lemma 5.

Theorem 5. In IS-algebra $(\mathrm{X}, *, \cdot, 0)$, The following conditions are equivalent:
(1) $AR(\mathrm{X})$ is ideal of X.
(2) $\forall a \in AR(\mathrm{X})$, $a_r : x \to x * a$ is injective.
(3) $\forall \upsilon \in \mathrm{X}$, $\nu \in SP(\mathrm{X})$,have $(\upsilon * \nu) * (0 * \nu) = \upsilon$.

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An identity related to Dedekind sums [1]

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Abstract The main purpose of this paper is using the analytic methods and the properties of the Dirichlet L-function to study a summation problem involving the Dedekind sums, and give an interesting identity for it.

Keywords Dedekind sums, Dirichlet series, identity.

§1. Introduction

For any integer $q \geq 2$ and integer h, the classical Dedekind sums is defined by

$$S(h,q) = \sum_{a=1}^{q} \left(\left(\frac{a}{q}\right)\right)\left(\left(\frac{ah}{q}\right)\right),$$

where

$$((x)) = \begin{cases} x - [x] - \frac{1}{2}, & \text{if } x \text{ is not an integer;} \\ 0, & \text{if } x \text{ is an integer.} \end{cases}$$

About the various properties of $S(h,q)$, many authors had studied it, and obtained a series results. For example, L.Carlitz [1] obtained a reciprocity theorem of $S(h,k)$. J. B. Conrey et al. [2] studied the mean value distribution of $S(h,k)$, and proved the following important asymptotic formula

$$\sum_{h=1}^{k}{}' |S(h,k)|^{2m} = f_m(k)\left(\frac{k}{12}\right)^{2m} + O\left(\left(k^{\frac{9}{5}} + k^{2m-1+\frac{1}{m+1}}\right)\log^3 k\right),$$

where $\sum_{h}'$ denotes the summation over all h such that $(k,h)=1$, and

$$\sum_{n=1}^{\infty} \frac{f_m(n)}{n^s} = 2\frac{\zeta^2(2m)}{\zeta(4m)} \cdot \frac{\zeta(s+4m-1)}{\zeta^2(s+2m)}\zeta(s).$$

Jia Chaohua [3] improved the error term in (1) as $O\left(k^{2m-1}\right)$, provide $m \geq 2$. Zhang Wenpeng [4] improved the error term of (1) for $m=1$. That is, he proved the following

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asymptotic formula

$$\sum_{h=1}^{k}{}' |S(h,k)|^2 = \frac{5}{144} k\phi(k) \cdot \frac{\prod_{p^\alpha \| k} \left(\left(1+\frac{1}{p}\right)^2 - \frac{1}{p^{3\alpha+1}} \right)}{\prod_{p|k} \left(1+\frac{1}{p}+\frac{1}{p^2}\right)} + O\left(k \exp\left(\frac{4\ln k}{\ln\ln k}\right)\right),$$

where $p^\alpha \| k$ denotes that $p^\alpha | k$ but $p^{\alpha+1} \dagger k$.

In this paper, we consider following Dirichlet series involving the Dedekind sums:

$$\sum_{a=1}^{\infty}{}' \sum_{b=1}^{\infty}{}' \frac{\mu(a)\mu(b)}{ab} S(a\bar{b}, q), \tag{1}$$

where $\sum_{a=1}^{\infty}{}'$ denotes the summation over all a with $(a,q)=1$, $\mu(a)$ be the Möbius function, and $\bar{b}$ denotes the solution of the congruent equation $xb \equiv 1 \pmod q$.

About the properties of (1), it seems that none had study it yet, at least we have not seen any related result before. In this paper, we use the analytic methods and the properties of Dirichlet L-function to study the calculating problem of (1), and give an interesting identity. That is, we shall prove the following:

Theorem. Let α be any positive integer. Then for any odd prime p, we have the identity

$$\sum_{a=1}^{\infty}{}' \sum_{b=1}^{\infty}{}' \frac{\mu(a)\mu(b)}{ab} S(a\bar{b}, p^\alpha) = \frac{p^\alpha}{2\pi^2} \left(1 + \frac{1}{p^2} + \frac{1}{p^4} + \cdots + \frac{1}{p^{2\alpha-2}}\right).$$

From this Theorem we may immediately deduce the following:

Corollary. For any odd prime p, we have the identity

$$\sum_{a=1}^{\infty}{}' \sum_{b=1}^{\infty}{}' \frac{\mu(a)\mu(b)}{ab} S(a\bar{b}, p) = \frac{p}{2\pi^2}.$$

For general positive integer $q \geq 3$, whether there exists a calculating formula for (1) is an open problem.

§2. Several Lemmas

To complete the proof of our Theorem, we need the following two simple Lemmas.

Lemma 1. Let $q \geq 3$ be a positive integer. Then for any integer c with $(c,q)=1$, we have the identity

$$S(c,\ q) = \frac{1}{\pi^2 q} \sum_{d|q} \frac{d^2}{\phi(d)} \sum_{\substack{\chi \bmod d \\ \chi(-1)=-1}} \chi(c) \mid L(1,\chi) \mid^2,$$

where $\phi(n)$ is the Euler function, $\sum_{\substack{\chi \bmod d \\ \chi(-1)=-1}}$ denotes the summation over all Dirichlet character modulo d with $\chi(-1)=-1$.

Proof. See Lemma 2 of reference [6].

Lemma 2. Let α be a positive integer, p be any odd prime. Then for any divisor $p^\beta > 1$ of p^α and any Dirichlet non-principal character χ modulo p^β, we have

$$L(1, \chi\chi^0_{p^\alpha}) = L(1, \chi),$$

where $\chi^0_{p^\alpha}$ denotes the principal character modulo p^α.

Proof. Let χ be any non-principal character modulo p^β. For any integer n with $(n, p^\beta) = 1$, it is clear that $(n, p^\alpha) = 1$. So from the Euler product formula (see reference [5]) we have

$$L(1, \chi\chi^0_{p^\alpha}) = \prod_q \left(1 - \frac{\chi(q)\chi^0_{p^\alpha}(q)}{q}\right) = \prod_q \left(1 - \frac{\chi(q)}{q}\right) = L(1, \chi),$$

where $\prod\limits_q$ denotes the product over all prime. This proves Lemma 2.

§3. Proof of Theorem

In this section, we shall use these two simple Lemmas to complete the proof of our Theorem. For any odd prime p and positive integer α, let $p^\beta \mid p^\alpha$ and $\beta \geq 1$, χ denotes any non-principal character modulo p^β. Note that

$$\frac{1}{L(1,\chi)} = \sum_{n=1}^\infty \frac{\mu(n)\chi(n)}{n},$$

where $\mu(n)$ denotes the Mӧbius function. From Lemma 1 and Lemma 2 we have

$$\begin{aligned}
& \sideset{}{'}\sum_{a=1}^\infty \sideset{}{'}\sum_{b=1}^\infty \frac{\mu(a)\mu(b)}{ab} S(a\bar{b}, p^\alpha) \\
= \quad & \frac{1}{\pi^2 p^\alpha} \sum_{d \mid p^\alpha} \frac{d^2}{\phi(d)} \sum_{\substack{\chi \bmod d \\ \chi(-1)=-1}} \sum_{\substack{a=1 \\ (ab,\ p)=1}}^\infty \sum_{b=1}^\infty \frac{\mu(a)\mu(b)\chi(a)\overline{\chi}(b)}{ab} \mid L(1,\chi) \mid^2 \\
= \quad & \frac{1}{\pi^2 p^\alpha} \sum_{d \mid p^\alpha} \frac{d^2}{\phi(d)} \sum_{\substack{\chi \bmod d \\ \chi(-1)=-1}} \frac{\mid L(1,\chi) \mid^2}{\mid L(1,\chi) \mid^2} \\
= \quad & \frac{1}{\pi^2 p^\alpha} \sum_{\substack{d \mid p^\alpha \\ d>1}} \frac{d^2}{\phi(d)} \sum_{\substack{\chi \bmod d \\ \chi(-1)=-1}} 1 = \frac{1}{\pi^2 p^\alpha} \sum_{\substack{d \mid p^\alpha \\ d>1}} \frac{d^2}{\phi(d)} \frac{\phi(d)}{2} \\
= \quad & \frac{p^\alpha}{2\pi^2} \left(1 + \frac{1}{p^2} + \frac{1}{p^4} + \cdots + \frac{1}{p^{2\alpha-2}}\right).
\end{aligned}$$

This completes the proof of our Theorem.

For general $q \geq 3$, from Lemma 1 and Lemma 2 we can deduce that

$$\begin{aligned}
&\sum_{a=1}^{\infty}{}' \sum_{b=1}^{\infty}{}' \frac{\mu(a)\mu(b)}{ab} S(a\bar{b}, q) \\
= \quad & \frac{1}{\pi^2 q} \sum_{d|q} \frac{d^2}{\phi(d)} \sum_{\substack{\chi \bmod d \\ \chi(-1)=-1}} \sum_{\substack{a=1 \\ (ab,\ q)=1}}^{\infty} \sum_{b=1}^{\infty} \frac{\mu(a)\mu(b)\chi(a)\overline{\chi}(b)}{ab} \mid L(1,\chi) \mid^2 \\
= \quad & \frac{1}{\pi^2 q} \sum_{d|q} \frac{d^2}{\phi(d)} \sum_{\substack{\chi \bmod d \\ \chi(-1)=-1}} \frac{\mid L(1,\chi) \mid^2}{\left| \sum_{r|q/d} \frac{\mu(r)\chi(r)}{r} \right|^2 \cdot \mid L(1,\chi) \mid^2} \\
= \quad & \frac{1}{\pi^2 q} \sum_{\substack{d|q \\ d>1}} \frac{d^2}{\phi(d)} \sum_{\substack{\chi \bmod d \\ \chi(-1)=-1}} \prod_{p|\frac{q}{d}} \left| 1 - \frac{\chi(p)}{p} \right|^{-2}.
\end{aligned}$$

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An equation involving the Smarandache dual function and Smarandache ceil function

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Abstract For any positive integer n, the Smarandache dual function $S^*(n)$ is defined as $S^*(n) = \max\{m: m \in N, m! \mid n\}$. For any fixed positive integer k and any positive integer n, the Smarandache ceil function $S_k(n)$ is defined as $S_k(n) = \min\{m: m \in N, n \mid m^k\}$. The main purpose of this paper is using the elementary methods to study the solvability of the equation $S^*(n) = S_k(n)$, and give its all positive integer solutions.

Keywords Smarandache dual function, Smarandache ceil function, equation, solution.

§1. Introduction and Results

For any positive integer n, the Smarandache dual function $S^*(n)$ is defined as the greatest positive m such that $m!$ divides n. That is,

$$S^*(n) = \max\{m: \quad m \in N, m! \mid n\}.$$

This function was introduced by J.Sandor in [1], where he studied the elementary properties of $S^*(n)$, and obtained a series interesting results .

On the other hand, for any fixed positive integer k and any positive integer n, the Smarandache ceil function $S_k(n)$ is defined as follows:

$$S_k(n) = \min\left\{m: \quad m \in N, n \mid m^k\right\}.$$

This function was introduced by F.Smarandache who proposed many problems in [2]. There are many papers on the Smarandache ceil function. For example, Ibstedt [3] and [4] studied these functions both theoretically and computationally, and got the following conclusions:

$$(\forall a, b \in N^*)\,(a, b) = 1 \Rightarrow S_k(ab) = S_k(a)S_k(b).$$

$$S_k(p_1^{\alpha_1}p_2^{\alpha_2}\dots p_r^{\alpha_r}) = S_k(p_1^{\alpha_1})S_k(p_2^{\alpha_2})\dots S_k(p_r^{\alpha_r}).$$

It is easily to show that $S_k(p^\alpha) = p^{\lceil \frac{\alpha}{k} \rceil}$, where p be a prime and $\lceil x \rceil$ denotes the least integer $\geq x$. So if $n = p_1^{\alpha_1}p_2^{\alpha_2}\dots p_r^{\alpha_r}$ is the factorization of n into prime power, then the following identity is obviously:

$$S_k(n) = S_k(p_1^{\alpha_1}p_2^{\alpha_2}\dots p_r^{\alpha_r}) = p_1^{\lceil \frac{\alpha_1}{k} \rceil}p_2^{\lceil \frac{\alpha_2}{k} \rceil}\dots p_r^{\lceil \frac{\alpha_r}{k} \rceil}.$$

In this paper, we use the elementary methods to study the solvability of the equation $S^*(n) = S_k(n)$, and give its all positive integer solutions. That is, we will prove the following:

Theorem. Let k be a fixed positive integer and n be any positive integer, then every positive integer solution of the equation $S^*(n) = S_k(n)$ can be expressed as

$$n = 1 \text{ or } (n,\ k) = (2^\alpha,\ k)\,,$$

where α be any positive integer, k be any integer $\geq \alpha$.

§2. Proof of the theorem

In this section, we will complete the proof of our Theorem. It is clear that $n = 1$ is a solution of the equation $S^*(n) = S_k(n)$ for any fixed positive integer k. Now we suppose that $n > 1$, we discuss the solutions of the equation $S^*(n) = S_k(n)$ in following several cases:

(I). If $n > 1$ be an odd integer, then from the definition of $S^*(n) = \max\{m:\ m \in N,\ m! \mid n\}$ and $2! \nmid n$ we have $S^*(n) = 1$. Now we discuss the function $S_k(n)$. Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_r^{\alpha_r}$ is the factorization of n into prime power, then we have

$$S_k(n) = S_k(p_1^{\alpha_1} p_2^{\alpha_2} \dots p_r^{\alpha_r}) = p_1^{\lceil \frac{\alpha_1}{k} \rceil} p_2^{\lceil \frac{\alpha_2}{k} \rceil} \dots p_r^{\lceil \frac{\alpha_r}{k} \rceil}.$$

Because n is an odd integer > 1, in the factorization $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_r^{\alpha_r}$, we have

(a). $\forall p_i$ is a prime ≥ 3;

(b). $\forall \alpha_i$ is an integer ≥ 0 and there exists at least an integer α_i satisfying $\alpha_i \geq 1$.

Then $\frac{\alpha_i}{k} \geq 0$. Hence $\lceil \frac{\alpha_i}{k} \rceil \geq 0$ and there exists at least an integer $\lceil \frac{\alpha_i}{k} \rceil \geq 1$. Therefore, $S_k(n)$ is an odd integer ≥ 3.

So in this case, $S^*(n) < S_k(n)$. This means that the equation $S^*(n) = S_k(n)$ has no odd positive integer solution if $n > 1$.

(II). If n be an even integer.

(a). If $n = 2^\alpha$, where α be any positive integer, then

$$S^*(n) = S^*(2^\alpha) = \max\{m:\ m \in N,\ m! \mid 2^\alpha\} = 2$$

and

$$S_k(n) = S_k(2^\alpha) = 2^{\lceil \frac{\alpha}{k} \rceil}.$$

It is obvious that $S^*(2^\alpha) = S_k(2^\alpha)$ if and only if $k \geq \alpha$, we have $0 < \frac{\alpha}{k} \leq 1$, then $\lceil \frac{\alpha}{k} \rceil = 1$. Only in this case it satisfy $S^*(n) = S_k(n) = 2$. So all integers $(n,\ k) = (2^\alpha,\ k)$ (where α be any positive integer and k be any integer $\geq \alpha$) are the solutions of the equation $S^*(n) = S_k(n)$.

(b). If $n = 2^\alpha \cdot m$, where $2 \nmid m$ and α be any positive integer, then we have

$$S_k(n) = S_k(2^\alpha \cdot m) = 2^{\lceil \frac{\alpha}{k} \rceil} \cdot S_k(m).$$

Let

$$u = S^*(n),$$

because m is an odd number, as is verified in (I), $S_k(m)$ is an odd integer ≥ 3. So $S_k(n) = 2^{\lceil \frac{\alpha}{k} \rceil} \cdot S_k(m)$ is even and it is impossible that $S_k(m) = 2^i$. Hence if $u = S^*(n) = S_k(n)$, u is not odd and $u \neq 2^i$. It just left one case, that is $u = 2^i \cdot l$, where $2 \nmid l$ and i be any positive integer.

If $u = 2^i \cdot l$ satisfying the equation $u = S^*(n) = S_k(n)$, from the definition of $S^*(n)$ we have $u! \mid n$. At the same time, from the definition of $S_k(n)$ we also have $n \mid u^k$.

So

$$u! \mid u^k.$$

It is a contradiction. Because if $u! \mid u^k$ is true, from

$$u - 1 \mid u!,$$

we get

$$u - 1 \mid u^k.$$

It is not true. Because $u = 2^i \cdot l \geq 6$, for any integer $u \geq 1$ we have

$$(u - 1, u) = 1.$$

Then

$$u - 1 \nmid u^k.$$

Therefore there is no solutions in this case.

Associating (I) and (II), we complete the proof of Theorem.

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On the quadratic mean value of the Smarandache dual function $S^{**}(n)$

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Abstract For any positive integer n, the Smarandache dual function $S^*(n)$ is defined as the greatest positive m such that $m!$ divides n. The Smarandache dual function $S^{**}(n)$ is defined as follows: $S^{**}(n)$ is the greatest positive $2m-1$ such that $(2m-1)!!$ divides n if n is an odd integer; $S^{**}(n)$ is the greatest positive $2m$ such that $(2m)!!$ divides n if n is an even integer. The main purpose of this paper is using the elementary methods to study the quadratic mean value of $S^{**}(n)$, and gives a more sharper asymptotic formulas for it.

Keywords Smarandache dual function, mean value, asymptotic formula.

§1. Introduction

For any positive integer n, the Smarandache dual function $S^*(n)$ is defined as the greatest positive m such that $m!$ divides n. That is,

$$S^*(n) = \max\{m : \ m \in N, \ m! \mid n\}.$$

Many people had studied the properties of $S^*(n)$, and obtained some interesting results, see references [1]-[4]. For example, J.Sandor [1] gave following propose: For any positive integer k, if q is the first prime following $2k+1$, then we have the identity

$$S^*((2k-1)!(2k+1)!) = q - 1.$$

Maohua Le [2] proved that this propose holds. Jie Li [3] studied the calculating problem of the series $\sum_{n=1}^{\infty} \frac{S^*(n)}{n^s}$, and proved following conclusions: For any real number $s > 1$, $\sum_{n=1}^{\infty} \frac{S^*(n)}{n^s}$ is absolutely convergent, and

$$\sum_{n=1}^{\infty} \frac{S^*(n)}{n^s} = \zeta(s) \sum_{n=1}^{\infty} \frac{1}{(n!)^s},$$

where $\zeta(s)$ is the Riemann zeta-function.

Shejiao Xue [4] studied the property of $\sum_{n=1}^{\infty} \frac{1}{SL^*(n)n^s}$, and gave the following identity

$$\sum_{n=1}^{\infty} \frac{1}{SL^*(n)n^s} = \zeta(s) \left(1 - \sum_{n=1}^{\infty} \frac{1}{n(n+1)((n+1)!)^s} \right),$$

where real number $s > 1$.

Su Gou and Xiaoying Du [5] defined the Smarandache LCM dual function $S^{**}(n)$ as follows:

$$S^{**}(n) = \begin{cases} \max\{2m-1: \ m \in N, \ (2m-1)!! \mid n\}, & 2 \dagger n, \\ \max\{2m: \ m \in N, \ (2m)!! \mid n\}, & 2|n. \end{cases}$$

and proved that : For any real number $s > 1$, $\displaystyle\sum_{n=1}^{\infty} \frac{S^{**}(n)}{n^s}$ is absolutely convergent, and

$$\sum_{n=1}^{\infty} \frac{S^{**}(n)}{n^s} = \zeta(s)\left(1 - \frac{1}{2^s}\right)\left(1 + \sum_{m=1}^{\infty} \frac{2}{((2m+1)!!)^s}\right) + \zeta(s)\sum_{m=1}^{\infty} \frac{2}{((2m)!!)^s}.$$

The main purpose of this paper is using the elementary methods to study the quadratic mean value properties of $S^{**}(n)$, and give a sharper asymptotic formulas for it. That is, we shall prove the following:

Theorem For any real integer $x \geq 1$, we have the asymptotic formula

$$\sum_{n \leq x} (S^{**}(n))^2 = \frac{13x}{2} + O\left(\left(\frac{\ln x}{\ln \ln x}\right)^3\right).$$

§2. Some Lemmas

In this section, we shall give two lemmas which are necessary in the proof of our theorem.

Lemma 1. Let $x \geq 1$ be a real number, for any positive integer k , if $(2k)!! \leq x < (2k+2)!!$, then we have the asymptotic formula

$$k = \frac{\ln x}{\ln \ln x} + O\left(\frac{\ln x \ln \ln \ln x}{(\ln \ln x)^2}\right).$$

Proof. For any positive integer k, we have

$$(2k)!! = 2^k k!, \ (2k+2)!! = 2^{k+1}(k+1)!.$$

If $(2k)!! \leq x < (2k+2)!!$, we know that

$$k \ln 2 + \sum_{i=1}^{k} \ln i \leq \ln x < (k+1)\ln 2 + \sum_{i=1}^{k+1} \ln i.$$

According to [6], we have

$$\sum_{i=1}^{k} \ln i = \left(k + \frac{1}{2}\right)\ln(k+1) - k - 1 + C + O\left(\frac{1}{k}\right) = k \ln k - k + O(\ln k).$$

$$\sum_{i=1}^{k+1} \ln i = \left(k + 1 + \frac{1}{2}\right)\ln(k+2) - k - 2 + C + O\left(\frac{1}{k}\right) = k \ln k - k + O(\ln k).$$

Thus

$$\ln x = k \ln k + k \ln 2 - k + O(\ln k).$$

Then

$$\begin{aligned}
\ln\ln x &= \ln\left[k\ln k + k\ln 2 - k + O(\ln k)\right]\\
&= \ln k + \ln\left[\ln k + \ln 2 - 1 + O\left(\frac{\ln k}{k}\right)\right]\\
&= \ln k + \ln\left\{\ln k\left[1 - \frac{1-\ln 2}{\ln k} + O\left(\frac{1}{k}\right)\right]\right\}\\
&= \ln k + \ln\ln k + \ln\left[1 - \frac{1-\ln 2}{\ln k} + O\left(\frac{1}{k}\right)\right]\\
&= \ln k - \ln\ln k + O\left(\frac{1}{\ln k}\right).
\end{aligned}$$

Thus

$$\ln k = \ln\ln x + O(\ln\ln k),$$

$$\ln\ln k = \ln\left\{\ln\ln x\left[1 + O\left(\frac{\ln\ln k}{\ln\ln x}\right)\right]\right\} = \ln\ln\ln x + O(1).$$

From above the asymptotic formula we may immediately deduce that

$$\begin{aligned}
k &= \frac{\ln x}{\ln k - 1 + \ln 2} + O\left(\frac{\ln k}{\ln k - 1 + \ln 2}\right)\\
&= \frac{\ln x}{\ln\ln x - \ln\ln k + O(\frac{1}{\ln k}) - 1 + \ln 2} + O\left(\frac{\ln k}{\ln k - 1 + \ln 2}\right)\\
&= \frac{\ln x}{\ln\ln x}\cdot\frac{1}{1 - \frac{\ln\ln k + 1 - \ln 2 + O(\frac{1}{\ln k})}{\ln\ln x}} + O\left(\frac{\ln k}{\ln k - 1 + \ln 2}\right)\\
&= \frac{\ln x}{\ln\ln x}\cdot\left[1 + \frac{\ln\ln k + 1 - \ln 2 + O\left(\frac{1}{\ln k}\right)}{\ln\ln x}\right] + O(1)\\
&= \frac{\ln x}{\ln\ln x} + O\left(\frac{\ln x\ln\ln\ln x}{(\ln\ln x)^2}\right).
\end{aligned}$$

This completes the proof of Lemma 1.

Lemma 2. Let $x \geq 1$ be a real number, for any positive integer k, if $(2k-1)!! \leq x < (2k+1)!!$, then we have the asymptotic formula

$$k = \frac{\ln x}{\ln\ln x} + O\left(\frac{\ln x\ln\ln\ln x}{(\ln\ln x)^2}\right).$$

Proof. For any positive integer k, we know that

$$(2k-1)!! = \frac{(2k)!}{(2k)!!} = \frac{(2k)!}{2^k k!},$$

$$(2k+1)!! = \frac{(2k+2)!}{2^{k+1}(k+1)!}.$$

Then

$$\begin{aligned}
\ln(2k-1)!! &= \sum_{i=1}^{2k}\ln i - k\ln 2 - \sum_{i=1}^{k}\ln i \\
&= \left(2k+\frac{1}{2}\right)\ln(2k+1) - 2k - 1 + C + O\left(\frac{1}{k}\right) - k\ln 2 \\
&- \left(k+\frac{1}{2}\right)\ln(k+1) + k + 1 - C + O\left(\frac{1}{k}\right) \\
&= k\ln k + k\ln 2 - k + O(\ln k)
\end{aligned}$$

Meanwhile

$$\begin{aligned}
\ln(2k+1)!! &= \sum_{i=1}^{2k+2}\ln i - (k+1)\ln 2 - \sum_{i=1}^{k+1}\ln i \\
&= \left(2k+2+\frac{1}{2}\right)\ln(2k+3) - 2k - 3 + C + O\left(\frac{1}{k}\right) - (k+1)\ln 2 \\
&- (k+1+\frac{1}{2})\ln(k+2) + k + 2 - C + O\left(\frac{1}{k}\right) \\
&= k\ln k + k\ln 2 - k + O(\ln k).
\end{aligned}$$

If $(2k-1)!! \leq x < (2k+1)!!$, we can get

$$\ln(2k-1)!! \leq \ln(x) < \ln(2k+1)!!.$$

Then we deduce that

$$\ln x = k\ln k + k\ln 2 - k + O(\ln k).$$

Now Lemma 2 follows from Lemma 1.

§3. Proof of the theorem

In this section, we shall complete the proof of the theorem. By the definition of $S^{**}(n)$, we know that: If $S^{**}(n) = 2m-1$, $n \leq x$, then $(2m-1)!!|n$. For $n = (2m-1)!!u$, $2m+1 \dagger u$

and $2 \dagger u$, $(2m-1)!! \leq x < (2m+1)!!$. By Lemma 1, we have

$$\begin{aligned}
\sum_{\substack{n \leq x \\ 2 \dagger n}} (S^{**}(n))^2 &= \sum_{\substack{(2m-1)!! u \leq x \\ 2m+1 \dagger u \quad 2 \dagger u}} (2m-1)^2 \\
&= \sum_{(2m-1)!! \leq x} (2m-1)^2 \sum_{\substack{u \leq \frac{x}{(2m-1)!!} \\ 2m+1 \dagger u \quad 2 \dagger u}} 1 \\
&= \sum_{(2m-1)!! \leq x} (2m-1)^2 \left[\frac{x}{2(2m-1)!!} - \frac{x}{2(2m+1)!!} \right] \\
&\quad + O\left(\sum_{(2m-1)!! \leq x} (2m-1)^2 \right) \\
&= \frac{x}{2} \sum_{(2m-1)!! \leq x} \left[\frac{(2m-1)^2}{(2m-1)!!} - \frac{(2m-1)^2}{(2m+1)!!} \right] + O\left(\sum_{(2m-1)!! \leq x} (2m-1)^2 \right) \\
&= \frac{x}{2} \sum_{m \leq \frac{\ln x}{\ln \ln x}} \left[\frac{(2m-1)^2}{(2m-1)!!} - \frac{(2m-1)^2}{(2m+1)!!} \right] + O\left(\left(\frac{\ln x}{\ln \ln x} \right)^2 \frac{\ln x \ln \ln \ln x}{(\ln \ln x)^2} \right) \\
&\quad + O\left(\sum_{m \leq \frac{\ln x}{\ln \ln x}} (2m-1)^2 \right) \\
&= \frac{x}{2} \sum_{m=1}^{\infty} \left[\frac{(2m-1)^2}{(2m-1)!!} - \frac{(2m-1)^2}{(2m+1)!!} \right] + O\left(x \sum_{(2m-1)!! > x} \frac{(2m-1)^2}{(2m-1)!!} \right) \\
&\quad + O\left(x \sum_{(2m-1)!! > x} \frac{(2m-1)^2}{(2m+1)!!} \right) + O\left(\left(\frac{\ln x}{\ln \ln x} \right)^3 \right) \\
&= \frac{x}{2} \sum_{m=1}^{\infty} \left[\frac{(2m-1)^2}{(2m-1)!!} - \frac{(2m-1)^2}{(2m+1)!!} \right] + O\left(\left(\frac{\ln x}{\ln \ln x} \right)^3 \right).
\end{aligned}$$

Then

$$\begin{aligned}
\sum_{m=1}^{\infty} \left[\frac{(2m-1)^2}{(2m-1)!!} - \frac{(2m-1)^2}{(2m+1)!!} \right] &= \sum_{m=1}^{\infty} \frac{(2m-1)^2}{(2m-1)!!} - \sum_{m=1}^{\infty} \frac{(2m-1)^2}{(2m+1)!!} \\
&= 1 + \sum_{m=1}^{\infty} \frac{(2m-1)^2 - (2m+1)^2}{(2m+1)!!} \\
&= 1 + 4 \sum_{m=1}^{\infty} \frac{2m}{(2m+1)!!} \\
&= 1 + 4 \sum_{m=1}^{\infty} \left[\frac{1}{(2m-1)!!} - \frac{1}{(2m+1)!!} \right] \\
&= 5.
\end{aligned}$$

So we obtain

$$\sum_{\substack{n\le x\\ 2\dagger n}} (S^{**}(n))^2 = \frac{5x}{2} + O\left(\left(\frac{\ln x}{\ln\ln x}\right)^3\right).$$

By the definition of $S^{**}(n)$, we know that: if $S^{**}(n) = 2m$, $n \le x$, then $(2m)!!|n$. let $n = (2m)!!v$, $2m+2 \dagger v$, then $(2m)!! \le x < (2m+2)!!$, By use Lemma 2, we have

$$\begin{aligned}
\sum_{\substack{n\le x\\ 2|n}} (S^{**}(n))^2 &= \sum_{\substack{(2m)!!v\le x\\ 2m+2\dagger v}} (2m)^2 \\
&= \sum_{(2m)!!\le x} (2m)^2 \sum_{\substack{v\le \frac{x}{(2m)!!}\\ 2m+2\dagger v}} 1 \\
&= \sum_{(2m)!!\le x} (2m)^2 \left[\frac{x}{(2m)!!} - \frac{x}{(2m+2)!!}\right] + O\left(\sum_{(2m)!!\le x} (2m)^2\right) \\
&= x \sum_{(2m)!!\le x} \left[\frac{(2m)^2}{(2m)!!} - \frac{(2m)^2}{(2m+2)!!}\right] + O\left(\sum_{(2m)!!\le x} (2m)^2\right) \\
&= x \sum_{m\le \frac{\ln x}{\ln\ln x}} \left[\frac{(2m)^2}{(2m)!!} - \frac{(2m)^2}{(2m+2)!!}\right] + O\left(\left(\frac{\ln x}{\ln\ln x}\right)^2 \frac{\ln x \ln\ln\ln x}{(\ln\ln x)^2}\right) \\
&\quad + O\left(\sum_{m\le \frac{\ln x}{\ln\ln x}} (2m)^2\right) \\
&= x \sum_{m=1}^{\infty} \left[\frac{(2m)^2}{(2m)!!} - \frac{(2m)^2}{(2m+2)!!}\right] + O\left(x \sum_{(2m)!!>x} \frac{(2m)^2}{(2m)!!}\right) \\
&+ O\left(x \sum_{(2m)!!>x} \frac{(2m)^2}{(2m+2)!!}\right) + O\left(\left(\frac{\ln x}{\ln\ln x}\right)^3\right) \\
&= x \sum_{m=1}^{\infty} \left[\frac{(2m)^2}{(2m)!!} - \frac{(2m)^2}{(2m+2)!!}\right] + O\left(\left(\frac{\ln x}{\ln\ln x}\right)^3\right).
\end{aligned}$$

Since

$$\begin{aligned}
\sum_{m=1}^{\infty} \left[\frac{(2m)^2}{(2m)!!} - \frac{(2m)^2}{(2m+2)!!}\right] &= \sum_{m=1}^{\infty} \frac{(2m)^2}{(2m)!!} - \sum_{m=1}^{\infty} \frac{(2m)^2}{(2m+2)!!} \\
&= 2 + \sum_{m=1}^{\infty} \frac{(2m+2)^2 - (2m)^2}{(2m+1)!!} \\
&= 2 + 4\sum_{m=1}^{\infty} \frac{2m+1}{(2m+2)!!} \\
&= 2 + 4\sum_{m=1}^{\infty} \left[\frac{1}{(2m)!!} - \frac{1}{(2m+2)!!}\right] \\
&= 4.
\end{aligned}$$

We have

$$\sum_{\substack{n\leq x\\ 2|n}} (S^{**}(n))^2 \quad = \quad 4x + O\left(\left(\frac{\ln x}{\ln\ln x}\right)^3\right).$$

By the above two estimates and the definition of $S^{**}(n)$, we have

$$\begin{aligned}\sum_{n\leq x} (S^{**}(n))^2 \quad &= \quad \sum_{\substack{n\leq x\\ 2\nmid n}} (S^{**}(n))^2 + \sum_{\substack{n\leq x\\ 2|n}} (S^{**}(n))^2 \\ &= \quad \frac{13x}{2} + O\left(\left(\frac{\ln x}{\ln\ln x}\right)^3\right).\end{aligned}$$

This completes the proof of Theorem.

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Distribution of quadratic residues over short intervals[1]

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Abstract Let p be an odd prime, $\mathcal{S}$ be the set of all quadratic residues mod p, n be a fixed positive integer with $(n,p)=1$ and $\ell|n$. Suppose $\mathcal{A}$ and $\mathcal{B}$ consist of consecutive integers coprime to p. Define

$$N(\mathcal{A},\mathcal{B},n,\ell;p)=\sum_{\substack{a\in\mathcal{A}\\(a+b,n)=\ell\\ab\in\mathcal{S}}}\sum_{b\in\mathcal{B}}1.$$

The main purpose of this paper is using the elementary methods to study the asymptotic properties of $N(\mathcal{A},\mathcal{B},n,\ell;p)$ and give a sharp asymptotic formula for it.

Keywords quadratic residue, short interval.

§1. Introduction

Let p be an odd prime. Suppose $\mathcal{A}$ and $\mathcal{B}$ consist of consecutive integers which are coprime to p, that is

$$\mathcal{A}=\{m:M<m\le M+A,(m,p)=1\},$$

$$\mathcal{B}=\{m:N<m\le N+B,(m,p)=1\},$$

where $A,B>0$. Let $\mathcal{S}$ be the set of all quadratic residues mod p, n be a fixed positive integer with $(n,p)=1$ and $\ell|n$. We consider the distribution of quadratic residues,

$$N(\mathcal{A},\mathcal{B},n,\ell;p)=\sum_{\substack{a\in\mathcal{A}\\(a+b,n)=\ell\\ab\in\mathcal{S}}}\sum_{b\in\mathcal{B}}1.$$

In this paper, we shall prove

Theorem 1.

$$N(\mathcal{A},\mathcal{B},n,\ell;p)=\frac{|\mathcal{A}||\mathcal{B}|}{2n}\phi\left(\frac{n}{\ell}\right)+E(\mathcal{A},\mathcal{B},n,\ell;p),$$

and

$$E(\mathcal{A},\mathcal{B},n,\ell;p)\ll\min\left((|\mathcal{A}||\mathcal{B}|)^{1/2},p\log^2p\right),$$

where $\phi(n)$ is the Euler function, $|\mathcal{A}|$ denotes the cardinality of $\mathcal{A}$, and the O-constant only depends on n and ℓ.

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Taking $n = 2$ and $\ell = 1$, we have

Corollary 1. The number of quadratic-tuples (a, b) in $\mathcal{A} \times \mathcal{B}$ such that a, b are of opposite parity and $a \cdot b$ is a quadratic residue mod p

$$= \frac{1}{4}|\mathcal{A}||\mathcal{B}| + O\left(\min\left((|\mathcal{A}||\mathcal{B}|)^{1/2}, p\log^2 p\right)\right).$$

Further, observing

$$\sum_{\ell|n}\ \sum_{(m,n)=\ell} = \sum_{m},$$

one can obtain that

Corollary 1. The number of quadratic-tuples (a, b) in $\mathcal{A} \times \mathcal{B}$ such that $a \cdot b$ is a quadratic residue mod p

$$= \sum_{\ell|n} N(\mathcal{A}, \mathcal{B}, n, \ell; p) = \frac{1}{2}|\mathcal{A}||\mathcal{B}| + O\left(\min\left((|\mathcal{A}||\mathcal{B}|)^{1/2}, p\log^2 p\right)\right).$$

§2. Proof of the theorem

In order to prove the theorem, we require a preliminary estimate, see reference [1], §5.1 Lemma 3.

Lemma 1. Let x be a real number, we denote $\|x\| = \min_{n\in\mathbb{Z}} |x - n|$. Assume U is a positive real number, K_0 is an integer, K is a positive integer, α and β are two arbitrary real numbers. If α can be written in the form

$$\alpha = \frac{h}{q} + \frac{\theta}{q^2}, \qquad (q, h) = 1,\ q \geq 1,\ |\theta| \leq 1,$$

we will have

$$\sum_{k=K_0+1}^{K_0+K} \min\left(U, \frac{1}{\|\alpha k + \beta\|}\right) \ll \left(\frac{K}{q} + 1\right)(U + q\log q).$$

Now we turn to prove the theorem. Obviously we have

$$\begin{aligned} N(\mathcal{A}, \mathcal{B}, n, \ell; p) &= \frac{1}{2}\sum_{\substack{a\in\mathcal{A}\\(a+b,n)=\ell}}\sum_{b\in\mathcal{B}}\left(1 + \left(\frac{ab}{p}\right)\right) \\ &= \frac{1}{2}\sum_{\substack{a\in\mathcal{A}\\(a+b,n)=\ell}}\sum_{b\in\mathcal{B}} 1 + \frac{1}{2}\sum_{\substack{a\in\mathcal{A}\\(a+b,n)=\ell}}\sum_{b\in\mathcal{B}}\left(\frac{ab}{p}\right) \\ &:= I_1 + I_2. \end{aligned} \tag{1}$$

Firstly,

$$\begin{aligned}
I_1 &= \frac{1}{2}\sum_{\substack{a\in\mathcal{A}\\(a+b,n)=\ell}}\sum_{b\in\mathcal{B}} 1 = \frac{1}{2}\sum_{a\in\mathcal{A}}\sum_{b\in\mathcal{B}}\sum_{r|\left(\frac{a+b}{\ell},\frac{n}{\ell}\right)}\mu(r)\\
&= \frac{1}{2}\sum_{a\in\mathcal{A}}\sum_{r|\frac{n}{\ell}}\mu(r)\sum_{\substack{b\in\mathcal{B}\\ b\equiv -a \pmod{r\ell}}} 1\\
&= \frac{1}{2}\sum_{a\in\mathcal{A}}\sum_{r|\frac{n}{\ell}}\mu(r)\left(\frac{|\mathcal{B}|}{r\ell}+O(1)\right)\\
&= \frac{|\mathcal{B}|}{2\ell}\sum_{a\in\mathcal{A}}\sum_{r|\frac{n}{\ell}}\frac{\mu(r)}{r}+O(|\mathcal{A}|)\\
&= \frac{|\mathcal{A}||\mathcal{B}|}{2n}\phi\left(\frac{n}{\ell}\right)+O(|\mathcal{A}|)\\
&= \frac{|\mathcal{A}||\mathcal{B}|}{2n}\phi\left(\frac{n}{\ell}\right)+O(\min(|\mathcal{A}|,|\mathcal{B}|)), \qquad (2)
\end{aligned}$$

and

$$\begin{aligned}
I_2 &= \frac{1}{2}\sum_{\substack{a\in\mathcal{A}\\(a+b,n)=\ell}}\sum_{b\in\mathcal{B}}\left(\frac{ab}{p}\right) = \frac{1}{2}\sum_{r|\frac{n}{\ell}}\mu(r)\sum_{\substack{a\in\mathcal{A}\\ r|\frac{a+b}{\ell}}}\sum_{b\in\mathcal{B}}\left(\frac{ab}{p}\right)\\
&= \frac{1}{2\ell}\sum_{r|\frac{n}{\ell}}\frac{\mu(r)}{r}\sum_{m=1}^{r\ell}\sum_{a\in\mathcal{A}}\sum_{b\in\mathcal{B}} e\left(\frac{m(a+b)}{r\ell}\right)\left(\frac{ab}{p}\right)\\
&= \frac{1}{2\ell}\sum_{r|\frac{n}{\ell}}\frac{\mu(r)}{r}\sum_{m=1}^{r\ell}\left(\sum_{a\in\mathcal{A}}\left(\frac{a}{p}\right)e\left(\frac{ma}{r\ell}\right)\right)\left(\sum_{b\in\mathcal{B}}\left(\frac{b}{p}\right)e\left(\frac{mb}{r\ell}\right)\right). \qquad (3)
\end{aligned}$$

On one hand, we define

$$T=\sum_{m=1}^{r\ell}\left(\sum_{a\in\mathcal{A}}\left(\frac{a}{p}\right)e\left(\frac{ma}{r\ell}\right)\right)\left(\sum_{b\in\mathcal{B}}\left(\frac{b}{p}\right)e\left(\frac{mb}{r\ell}\right)\right).$$

Applying Cauchy Inequality, we have

$$\begin{aligned}
T^2 &\leq \sum_{m=1}^{r\ell}\left|\left(\sum_{a\in\mathcal{A}}\left(\frac{a}{p}\right)e\left(\frac{ma}{r\ell}\right)\right)\right|^2\times\sum_{n=1}^{r\ell}\left|\left(\sum_{b\in\mathcal{B}}\left(\frac{b}{p}\right)e\left(\frac{nb}{r\ell}\right)\right)\right|^2\\
&= r^2\ell^2|\mathcal{A}||\mathcal{B}|.
\end{aligned}$$

Thus

$$I_2 \ll_{n,\ell} (|\mathcal{A}||\mathcal{B}|)^{1/2}. \qquad (4)$$

On the other hand, we define

$$G_1=\sum_{a\in\mathcal{A}}\left(\frac{a}{p}\right)e\left(\frac{ma}{r\ell}\right), G_2=\sum_{b\in\mathcal{B}}\left(\frac{b}{p}\right)e\left(\frac{mb}{r\ell}\right).$$

It suffices to estimate G_1 and G_2 separately.

Notice that

$$\left(\frac{a}{p}\right) = \frac{1}{p}\sum_{s=1}^{p-1} G(s)e\left(-\frac{as}{p}\right),$$

where $G(s) = \sum_{n=1}^{p-1}\left(\frac{n}{p}\right) e\left(\frac{ns}{p}\right)$, and also notice that $\frac{m}{r\ell} - \frac{s}{p} \neq 0$ for $1 \le m \le r\ell, 1 \le s \le p-1$ and $(r\ell, p) = 1$, thus

$$\begin{aligned} G_1 &= \frac{1}{p}\sum_{a\in\mathcal{A}}\sum_{s=1}^{p-1} G(s)e\left(\left(\frac{m}{r\ell} - \frac{s}{p}\right)a\right) = \frac{1}{p}\sum_{s=1}^{p-1} G(s)\frac{f(\mathcal{A}, m, s; r, \ell, p)}{e\left(\frac{s}{p} - \frac{m}{r\ell}\right) - 1} \\ &\ll \frac{1}{\sqrt{p}}\sum_{s=1}^{p-1}\frac{1}{\left|e\left(\frac{s}{p} - \frac{m}{r\ell}\right) - 1\right|} \ll \frac{1}{\sqrt{p}}\sum_{s=1}^{p-1}\frac{1}{\left\|\frac{s}{p} - \frac{m}{r\ell}\right\|} \end{aligned}$$

Since

$$\left\|\frac{s}{p} - \frac{m}{r\ell}\right\| = \left\|\frac{sr\ell - mp}{pr\ell}\right\| \ge \frac{1}{pr\ell},$$

Lemma 1 yields that

$$G_1 \ll \frac{1}{\sqrt{p}}\sum_{s\le p-1}\min\left(pr\ell, \frac{1}{\left\|\frac{s}{p} - \frac{m}{r\ell}\right\|}\right) \ll \sqrt{p}\log p. \tag{5}$$

Similarly,

$$G_2 \ll \sqrt{p}\log p. \tag{6}$$

Combining (3), (5) and (6), we have

$$I_2 \ll p\log^2 p, \tag{7}$$

where the O constant depends on n and ℓ.

The theorem follows immediately from (1), (2), (4) and (7).

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An equation involving the Smarandache power function

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Abstract For any positive integer n, the famous Smarandache power function $SP(n)$ is defined as the smallest positive integer m such that m^m is divided by n. That is, $SP(n) = \min\{m : n \mid m^m,\ m \in N \prod_{p|m} p = \prod_{p|n} p\ \}$. The main purpose of this paper is using the elementary and analytic methods to study the solvability of the equation $SP(1) + SP(2) + \cdots + SP(n) = SP(\frac{n(n+1)}{2})$, and give its all positive integer solutions $n = 1,\ 2,\ 3$.

Keywords The Smarandache power function, Möbius function, positive integer solutions.

§1. Introduction

For any positive integer n, the famous Smarandache power function $SP(n)$ is defined as the smallest positive integer m such that m^m is divided by n. That is, $SP(n) = \min\{m : n \mid m^m,\ m \in N \prod_{p|m} p = \prod_{p|n} p\ \}$. For example, the first few values of $SP(n)$ are: $SP(1) = 1$, $SP(2) = 2$, $SP(3) = 3$, $SP(4) = 2$, $SP(5) = 5$, $SP(6) = 6$, $SP(7) = 7$, $SP(8) = 4$, $SP(9) = 3$, $SP(10) = 10$, $SP(11) = 11$, $SP(12) = 6$, $SP(13) = 13$, $SP(14) = 14$, $SP(15) = 15$, $SP(16) = 4$, $SP(17) = 17$, $SP(18) = 6$, $SP(19) = 19$, $SP(20) = 10$, $\cdots$.

In reference [1], Professor F.Smarandache asked us to study the properties of $SP(n)$. From the definition of $SP(n)$ we can easily get the following conclusions: If $n = p^\alpha$, then

$$SP(n) = \begin{cases} p, & 1 \le \alpha \le p; \\ p^2, & p+1 \le \alpha \le 2p^2; \\ p^3, & 2p^2+1 \le \alpha \le 3p^3; \\ \cdots & \cdots \\ p^\alpha, & (\alpha-1)p^{\alpha-1}+1 \le \alpha \le \alpha p^\alpha. \end{cases}$$

Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$ denotes the factorization of n into prime powers. If $\alpha_i \le p_i$ for all α_i $(i = 1,\ 2,\ \cdots,\ r)$, then we have $SP(n) = U(n)$, where $U(n) = \prod_{p|n} p$, $\prod_{p|n}$ denotes the product over all different prime divisors of n. It is clear that $SP(n)$ is not a multiplicative function. For example, $SP(3) = 3$, $SP(8) = 4$, $SP(24) = 6 \neq SP(3) \times SP(8)$. But for most n we have $SP(n) = U(n)$.

About the deeply arithmetical properties of $SP(n)$, many scholars had studied it, and obtained a series results. See [2], [3] and [4]. For example, Chenglian Tian [2] used the elementary

method to study the solvability of the equation $SP(n^k) = \phi(n)$, and give its all positive integer solutions for $k = 1, 2, 3$. Zhefeng Xu [3] studied the mean value properties of $SP(n)$, and obtained an interesting asymptotic formula for $\sum\limits_{n\le x} SP(n)$.

The main purpose of this paper is using the elementary and analytic methods to study the solvability of the equation $SP(1) + SP(2) + \cdots, +SP(n) = SP(\frac{n(n+1)}{2})$, and give its all positive solutions. That is, we shall prove the following:

Theorem. For any positive integer n, the equation

$$SP(1) + SP(2) + \cdots + SP(n) = SP\left(\frac{n(n+1)}{2}\right) \tag{2}$$

holds if and only if $n = 1, 2, 3$.

§2. A Lemma.

To complete the proof of Theorem, we need the following:

Lemma. For any positive integer $n \ge 1500$, we have the inequality

$$SP(1) + SP(2) + \cdots + SP(n) \ge \frac{n(n+1)}{4}.$$

Proof. Let A denotes the set of all square-free numbers. That is, $A = \{n :\ n = p_1p_2\cdots p_r,$ where p_i are deferent primes.$\}$. Note that $|\mu(n)| = \sum_{d^2|n}\mu(d)$, then from the properties of the Mӧbius function we have

$$\begin{aligned}
& SP(1) + SP(2) + \cdots + SP(n) \\
\ge\ & \sum_{\substack{a\le n\\ a\in A}} SP(a) = \sum_{\substack{a\le n\\ a\in A}} a = \sum_{a\le n} a \mid \mu(a) \mid = \sum_{a\le n}\sum_{d^2|a} a\mu(d) \\
=\ & \sum_{d^2a\le n} d^2 a\mu(d) = \sum_{d^2\le n} d^2\mu(d) \sum_{a\le \frac{n}{d^2}} a \\
=\ & \sum_{d\le\sqrt{n}} d^2\mu(d)\frac{\left[\frac{n}{d^2}\right]\left(\left[\frac{n}{d^2}\right]+1\right)}{2} \\
=\ & \sum_{d\le\sqrt{n}} d^2\mu(d)\left\{\frac{n^2}{2d^4} - \frac{n}{d^2}\left\{\frac{n}{d^2}\right\} + \frac{n}{2d^2} + \frac{1}{2}\left\{\frac{n}{d^2}\right\}^2 - \frac{1}{2}\left\{\frac{n}{d^2}\right\}\right\} \\
=\ & \frac{n^2}{2}\sum_{d\le\sqrt{n}}\frac{\mu(d)}{d^2} - n\sum_{d\le\sqrt{n}}\mu(d)\left\{\frac{n}{d^2}\right\} + \frac{n}{2}\sum_{d\le\sqrt{n}}\mu(d) \\
+\ & \frac{1}{2}\sum_{d\le\sqrt{n}} d^2\mu(d)\left\{\frac{n}{d^2}\right\}^2 - \frac{1}{2}\sum_{d\le\sqrt{n}} d^2\mu(d)\left\{\frac{n}{d^2}\right\} \\
\ge\ & \frac{n^2}{2}\sum_{d=1}^{\infty}\frac{\mu(d)}{d^2} - \frac{n^2}{2}\sum_{d>\sqrt{n}}\frac{\mu(d)}{d^2} - n\sqrt{n} - \frac{n\sqrt{n}}{2} - \frac{n\sqrt{n}}{2} - \frac{n\sqrt{n}}{2} \\
=\ & \frac{n^2}{2}\frac{1}{\zeta(2)} - \frac{5}{2}n^{\frac{3}{2}} = \frac{n^2}{2}\frac{6}{\pi^2} - \frac{5}{2}n^{\frac{3}{2}} = \frac{3n^2}{\pi^2} - \frac{5}{2}n^{\frac{3}{2}}.
\end{aligned}$$

For any positive integer $n \geq 1500$, we can easily get the inequality

$$\frac{3n^2}{\pi^2} - \frac{5}{2}n^{\frac{3}{2}} > \frac{n^2 + n^{\frac{3}{2}}}{4} > \frac{n(n+1)}{4}.$$

This completes the proof of Lemma.

§3. Proof of the theorem

In this section, we shall use the above Lemma to complete the proof of our Theorem. In fact from the definition of the function $SP(n)$ we can easily deduce that $n = 1,\ 2,\ 3$ are the positive integer solutions of the equation (2). Now we shall prove that for any positive integer $n \geq 4$, the equation (2) does not hold. To prove this, we consider following two cases:

(A). For any positive integer $n \geq 4$, if $\frac{n(n+1)}{2} = p_1 p_2 \cdots p_r$, where $p_i,\ (i = 1,\ 2,\ \cdots,\ r)$ are deferent primes, then we have

$$SP(1) + SP(2) + \cdots + SP(n) < \frac{n(n+1)}{2} = p_1 p_2 \cdots p_r = SP\left(\frac{n(n+1)}{2}\right).$$

(B). For any positive integer $n \geq 4$, if $\frac{n(n+1)}{2} = {p_1}^{\alpha_1} {p_2}^{\alpha_2} \cdots {p_r}^{\alpha_r}$, let $\alpha = \max\{\alpha_1,\ \alpha_2,\ \cdots \alpha_r\}$ and $\alpha \geq 2$, then we have

$$SP\left(\frac{n(n+1)}{2}\right) \leq \frac{n(n+1)}{4}. \tag{3}$$

Combining the above section Lemma and (3) we may immediately deduce the inequality

$$SP(1) + SP(2) + \cdots + SP(n) > \frac{n(n+1)}{4} > SP\left(\frac{n(n+1)}{2}\right),$$

if $n \geq 1500$. So from this inequality we know that there is no positive integer n satisfying the equation (2), if $n \geq 1500$.

Combining the above all conclusions we know that the equation (2) has and only has three positive integer solutions $n = 1,\ 2,\ 3$. This completes the proof of our Theorem.

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A new Smarandache multiplicative function and its mean value formula

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Abstract For any positive integer n, we define the Smarandache multiplicative function $D(n)$ as the smallest positive integer m such that n divide $d(1)d(2)\cdots d(m)$, where $d(n)$ is the Dirichlet divisor function. The main purpose of this paper is using the elementary and analytic method to study the mean value properties of $\ln(D(n))$, and give two interesting asymptotic formula.

Keywords Smarandache multiplicative function, mean value, asymptotic formula.

§1. Introduction and results

For any positive integer n, the famous Dirichlet divisor function $d(n)$ is defined as the number of all distinct positive divisors of n. If $n = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdots p_r^{\alpha_r}$ be the prime power factorization of n, then from the definition and properties of $d(n)$ we may get

$$d(n) = (\alpha_1 + 1) \cdot (\alpha_2 + 1) \cdots (\alpha_r + 1).$$

From this formula we may immediately deduce that the first few values of $d(n)$ are $d(1) = 1$, $d(2) = 2$, $d(3) = 2$, $d(4) = 3$, $d(5) = 2$, $d(6) = 4$, $d(7) = 2$, $d(8) = 4$ $d(9) = 3$, $d(10) = 4$, $d(11) = 2$, $d(12) = 6$, $d(13) = 2$, $d(14) = 4$, $d(15) = 4$, $d(16) = 5$, $d(17) = 2$, $d(18) = 6$, $d(19) = 2$, $d(20) = 6$, $\cdots$. Now for any positive integer n, we use divisor function $d(n)$ to define a new number theory function $D(n)$ as follows:

$D(n)$ denotes the smallest positive integer m such that n divide product $d(1)d(2)\cdots d(m)$. That is,

$$D(n) = \min\left\{ m : \ n | \prod_{i=1}^{m} d(i) \right\}.$$

For example, $D(1) = 1$, $D(2) = 2$, $D(3) = 4$, $D(4) = 3$, $D(5) = 2^4$, $D(6) = 4$, $D(7) = 2^6$, $D(8) = 5$, $D(9) = 9$, $D(10) = 16$, $D(11) = 2^{10}$, $D(12) = 4$, $D(13) = 2^{12}$, $D(14) = 64$, $D(15) = 16$, $D(16) = 6$, $D(17) = 2^{16}$, $D(18) = 9 \cdots$. Recently, Professor Zhang Wenpeng asked us to study the mean value properties of $\ln(D(n))$. About this problem, it seems that none had studied it yet, at least we have not seen any related papers before. I think this

problem is interesting, because there are some close relations between $D(n)$ and the Dirichlet divisor function $d(n)$, so it can help us to find more information about $d(n)$. The main purpose of this paper is using the elementary method to study the mean value properties of $\ln(D(n))$ and $\frac{\ln(D(n))}{n}$, and give two interesting asymptotic formulae for them. That is, we shall prove the following:

Theorem 1. For any real number $x \geq 2$, we have the asymptotic formula

$$\sum_{n\leq x} \ln(D(n)) = \frac{\pi^2 \cdot \ln 2}{12} \cdot \frac{x^2}{\ln x} + \sum_{i=2}^{k} \frac{c_i \cdot x^2}{\ln^i x} + O\left(\frac{x^2}{\ln^{k+1} x}\right),$$

where c_i $(i = 2,\ 3,\ \cdots,\ k)$ are computable constants.

Theorem 2. For any real number $x \geq 2$, we also have the asymptotic formula

$$\sum_{n\leq x} \frac{\ln(D(n))}{n} = \frac{\pi^2 \cdot \ln 2}{6} \cdot \frac{x}{\ln x} + \sum_{i=2}^{k} \frac{d_i \cdot x}{\ln^i x} + O\left(\frac{x}{\ln^{k+1} x}\right),$$

where d_i $(i = 2,\ 3,\ \cdots,\ k)$ are computable constants.

§2. Some Lemmas.

In this Section, we shall give two simple Lemmas which are necessary in the proof of our Theorems. First we have the following:

Lemma 1. For any prime p, we have the identity $D(p) = 2^{p-1}$.

Proof. In fact for any prime p, let $D(p) = m$, from the definition of $D(n)$ we have p divide $d(1)d(2)\cdots d(m)$, so p divide one of $d(1),\ d(2),\ \cdots,\ d(m)$. Since m is the smallest positive integer such that p divide $d(1)d(2)\cdots d(m)$, so $p \mid d(m)$ and $m = 2^{p-1}$.

Lemma 2. For any positive integer $n \geq 2$, $D(n)$ is a Smarandache multiplicative function.

Proof. From the definition of $D(n)$ we can easily deduce this conclusion.

Lemma 3. Let p denotes a prime, then we have the asymptotic formula

$$\sum_{2\leq p\leq \frac{x}{m}} p = \frac{x^2}{2m^2 \cdot \ln \frac{x}{m}} + \sum_{i=2}^{k} \frac{b_i \cdot x^2}{m^2 \cdot \ln^i \frac{x}{m}} + O\left(\frac{x^2}{m^2 \cdot \ln^{k+1} \frac{x}{m}}\right),$$

where b_i $(i = 2,\ 3\ldots,\ k)$ are computable constants.

Proof. Let $\pi(x)$ denotes the number of all primes not to exceeding x, note that for any positive integer k, we have the asymptotic formula (see reference [7])

$$\pi(x) = \sum_{i=1}^{k} \frac{a_i \cdot x}{\ln^i x} + O\left(\frac{x}{\ln^{k+1} x}\right),$$

where a_i $(i = 1,\ 2,\ \cdots,\ k)$ are constants and $a_1 = 1$.

Using the above asymptotic formula, the Able's Summation formula (see reference [6]) and

Prime Theorem (see reference [7]) we have

$$\begin{aligned}\sum_{2\le p\le \frac{x}{m}} p &= \pi\left(\frac{x}{m}\right)\cdot\frac{x}{m}-\pi(2)\cdot 2-\int_2^{\frac{x}{m}}\pi(t)dt\\ &= \frac{x^2}{2m^2\cdot\ln\frac{x}{m}}+\sum_{i=2}^{k}\frac{b_i\cdot x^2}{m^2\cdot\ln^i\frac{x}{m}}+O\left(\frac{x^2}{m^2\cdot\ln^{k+1}\frac{x}{m}}\right),\end{aligned}$$

where b_i $(i=2,\ 3\ldots,\ k)$ are computable constants.

§2. Proof of the theorems

In this section, we shall complete the proof of our Theorems. First we prove Theorem 1. We define two sets A and B as following:

$$A=\{n:\ n\le x,\ P(n)<\sqrt{n}\}$$

and

$$B=\{n:\ n\le x,\ P(n)\ge\sqrt{n}\},$$

where $P(n)$ denotes the biggest prime divisor of n. By Lemma 1 and Lemma 2 we have

$$\sum_{n\in A}\ln D(n)\le\sum_{n\le x}\ln 2^{\sqrt{n}}\ln n=\ln 2\cdot\sum_{n\le x}\sqrt{n}\ln n\le\ln 2\cdot x^{\frac{3}{2}}\ln x\ll x^{\frac{3}{2}}\ln x. \tag{1}$$

$$\begin{aligned}\sum_{n\in B}\ln D(n) &= \sum_{\substack{n\le x\\ P(n)\ge\sqrt{n}}}\ln D(n)=\sum_{\substack{mp\le x\\ p>m}}\ln(D(p))=\sum_{\substack{mp\le x\\ p>m}}\ln 2^{p-1}=\ln 2\cdot\sum_{\substack{mp\le x\\ p>m}}(p-1)\\ &= \ln 2\cdot\sum_{m\le\sqrt{x}}\sum_{m<p\le\frac{x}{m}}(p-1)=\ln 2\cdot\left(\sum_{m\le\sqrt{x}}\sum_{m<p\le\frac{x}{m}}p-\sum_{m\le\sqrt{x}}\sum_{m<p\le\frac{x}{m}}1\right).\end{aligned} \tag{2}$$

It is clear that

$$\sum_{m\le\sqrt{x}}\sum_{m<p\le\frac{x}{m}}1=\sum_{m\le\sqrt{x}}\frac{x}{m}=x\cdot\sum_{m\le\sqrt{x}}\frac{1}{m}=x\cdot\left(\log\sqrt{x}+c+O\left(\frac{1}{\sqrt{x}}\right)\right)<x^{\frac{3}{2}}. \tag{3}$$

Using Lemma 3 we have the asymptotic formula

$$\begin{aligned}\sum_{m\le\sqrt{x}}\sum_{m<p\le\frac{x}{m}}p &= \sum_{m\le\sqrt{x}}\left(\frac{x^2}{2m^2\cdot\ln\frac{x}{m}}+\sum_{i=2}^{k}\frac{b_i\cdot x^2}{m^2\cdot\ln^i\frac{x}{m}}+O\left(\frac{x^2}{m^2\cdot\ln^{k+1}\frac{x}{m}}\right)\right)\\ &= \sum_{m\le\ln^2 x}\left(\frac{x^2}{2m^2\cdot\ln\frac{x}{m}}+\sum_{i=2}^{k}\frac{b_i\cdot x^2}{m^2\cdot\ln^i\frac{x}{m}}+O\left(\frac{x^2}{m^2\cdot\ln^{k+1}\frac{x}{m}}\right)\right)\\ &\quad+\sum_{\ln^2 x\le m\le\sqrt{x}}\left(\frac{x^2}{2m^2\cdot\ln\frac{x}{m}}+\sum_{i=2}^{k}\frac{b_i\cdot x^2}{m^2\cdot\ln^i\frac{x}{m}}+O\left(\frac{x^2}{m^2\cdot\ln^{k+1}\frac{x}{m}}\right)\right)\\ &= \frac{\pi^2\cdot\ln 2}{12}\cdot\frac{x^2}{\ln x}+\sum_{i=2}^{k}\frac{c_i\cdot x^2}{\ln^i x}+O\left(\frac{x^2}{\ln^{k+1}x}\right).\end{aligned} \tag{4}$$

Combining(1), (2), (3) and (4) we may immediately deduce that

$$\sum_{n\le x}\ln(D(n))=\frac{\pi^2\cdot\ln 2}{12}\cdot\frac{x^2}{\ln x}+\sum_{i=2}^{k}\frac{c_i\cdot x^2}{\ln^i x}+O\left(\frac{x^2}{\ln^{k+1}x}\right),$$

where c_i $(i=2,\ 3,\ \ldots,\ k)$ are computable constants. This proves Theorem 1.

Now we prove Theorem 2. From the Abel's summation formula and Theorem 1 we have

$$\begin{aligned}\sum_{n\le x}\frac{\ln(D(n))}{n} &= \frac{1}{x}\cdot\sum_{n\le x}\ln(D(n))+\int_2^x\frac{1}{t^2}\sum_{n\le t}\ln(D(t))dt\\
&= \frac{1}{x}\left(\frac{\pi^2\cdot\ln 2}{12}\cdot\frac{x^2}{\ln x}+\sum_{i=2}^{k}\frac{c_i\cdot x^2}{\ln^i x}+O\left(\frac{x^2}{\ln^{k+1}x}\right)\right)\\
&\quad+\int_2^x\frac{1}{t^2}\left(\frac{\pi^2\cdot\ln 2}{12}\cdot\frac{t^2}{\ln t}+\sum_{i=2}^{k}\frac{c_i\cdot t^2}{\ln^i t}+O\left(\frac{t^2}{\ln^{k+1}t}\right)\right)dt\\
&= \frac{\pi^2\cdot\ln 2}{12}\cdot\frac{x}{\ln x}+\sum_{i=2}^{k}\frac{c_i\cdot x}{\ln^i x}+O\left(\frac{x}{\ln^{k+1}x}\right)\\
&\quad+\frac{\pi^2\cdot\ln 2}{12}\int_2^x\frac{1}{\ln t}dt+\sum_{i=2}^{x}c_i\int_2^x\frac{1}{\ln^i t}dt+O\left(\int_2^x\frac{1}{\ln^{k+1}t}dt\right)\\
&= \frac{\pi^2\cdot\ln 2}{6}\cdot\frac{x}{\ln x}+\sum_{i=2}^{k}\frac{d_i\cdot x}{\ln^i x}+O\left(\frac{x}{\ln^{k+1}x}\right),\end{aligned}$$

where d_i $(i=2,\ 3,\ \ldots,\ k)$ are computable constants.

This completes the proof of Theorem 2.

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The mean value of the k-th Smarandache dual function

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Abstract The main purpose of this paper is using the elementary and analytic methods to study the mean value properties of the k-th Smarandache dual function, and give a sharper asymptotic formula for it.

Keywords Riemann zeta-function, Perron's formula, the k-th Smarandache dual function.

§1. Introduction

Let k be a fixed positive integer. For any positive integer n, the F.Smarandache ceil function of order k is defined as the smallest positive integer m such that n divide m^k. That is, $S_k(n) = \min\{m: \ m \in N, \ n|m^k\}$. The dual function of this function $\overline{S_k}(n)$ is defined as the largest positive integer m such that m^k divide n. That is, $\overline{S_k}(n) = \max\{m: \ m \in N, m^k|n\}$. For examples, if $k = 2$, then the first few value of $S_2(n)$ are $S_2(1) = 1, \ S_2(2) = 2, \ S_2(3) = 3, \ S_2(4) = 2, \ S_2(5) = 5, \ S_2(6) = 6, \ S_2(7) = 7, \ S_2(8) = 4, \ S_2(9) = 3, \cdots$. The first few value of $\overline{S_2}(n)$ are $\overline{S_2}(1) = 1, \ \overline{S_2}(2) = 1, \ \overline{S_2}(3) = 1, \ \overline{S_2}(4) = 2, \ \overline{S_2}(5) = 1, \ \overline{S_k}(6) = 1, \ \overline{S_k}(7) = 1, \ \overline{S_k}(8) = 2, \ \overline{S_k}(9) = 3, \cdots$.

About the properties of functions $S_k(n)$ and $\overline{S_k}(n)$, many scholars have studied them, and obtained some interesting conclusions, see references [2], [3], [4] and [5]. For example, Wang Yongxing [2] proved that for any positive integers a and b with $(a, \ b) = 1$, we have

$$\overline{S_k}(ab) = \max\{m: \ m \in N, \ m^k|a\} \cdot \max\{m: \ m \in N, \ m^k|b\} = \overline{S_k}(a) \cdot \overline{S_k}(b)$$

and

$$\overline{S_k}\,(p^\alpha) = p^{\lfloor \frac{\alpha}{k} \rfloor},$$

where $\lfloor x \rfloor$ denotes the smallest positive integer $\geq x$. For any positive integer n, if $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$ denotes the factorization of n into prime powers, then we can deduce the identity

$$\overline{S_k}(p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}) = p_1^{\lfloor \frac{\alpha_1}{k} \rfloor} p_2^{\lfloor \frac{\alpha_2}{k} \rfloor} \cdots p_r^{\lfloor \frac{\alpha_r}{k} \rfloor} = \overline{S_k}(p_1^{\alpha_1}) \overline{S_k}(p_2^{\alpha_2}) \cdots \overline{S_k}(p_r^{\alpha_r}).$$

From this properties we know that $\overline{S_k}(n)$ is a multiplicative function, so we can use the Euler product formula and the analytic method to study the mean value properties of $\overline{S_k}(n)$, and obtain an interesting mean value formula for it. The main purpose of this paper is using the elementary and analytic methods to study this problem, and prove the following conclusion:

Theorem. For any real number $x \geq 1$, we have the asymptotic formula

$$\sum_{n\leq x}\overline{S_k}(n)=\begin{cases}\frac{3}{\pi^2}\left(\ln x+3\gamma-1-\frac{12}{\pi^2}\zeta'(2)\right)\cdot x+O\left(x^{\frac{3}{4}}\cdot\ln x\right), & \text{if } k=2;\\ \frac{\zeta(k-1)}{\zeta(k)}x+O\left(\min\{x^{\frac{2}{k}},\ x^{\frac{1}{2}+\epsilon}\}\right), & \text{if } k>2.\end{cases}$$

where $\zeta(s)$ is the Riemann zeta-function, ϵ denotes any fixed positive number, γ is the Euler constant and $\zeta'(2)=-\sum\limits_{n=1}^{\infty}\frac{\ln n}{n^2}$.

§2. Proof of the theorem

In this section, we shall complete the proof of our Theorem. First we give three simple Lemmas which are necessary in the proof of our Theorem. The proofs of these Lemmas can be found in reference [8].

Lemma 1. For any real number $x \geq 1$ and $\alpha > 0$, we have the asymptotic formula

$$\sum_{n\leq x} n^{\alpha}=\frac{x^{1+\alpha}}{1+\alpha}+O\left(x^{\alpha}\right).$$

Lemma 2. For any real number $x \geq 1$, we have the asymptotic formula

$$\sum_{n\leq x}\frac{1}{n}=\log x+\gamma+O\left(\frac{1}{x}\right),$$

where γ is the Euler constant.

Lemma 3. For any real number $x \geq 1$ and $1 \neq s > 0$, we have the asymptotic formula

$$\sum_{n\leq x}\frac{1}{n^s}=\frac{x^{1-s}}{1-s}+\zeta(s)+O\left(x^{-s}\right),$$

where $\zeta(s)$ is the Riemann zeta-function.

Now we use these Lemmas to prove our conclusion. First we use the elementary method to obtain an asymptotic formula. From the definition of $\overline{S_k}(n)$ we can assume that $\overline{S_k}(n)=m$ and $n=m^kh$, where h is a k-th power-free number. That is, for any prime p, p^k does not divide h. Note that if h is a k-th power free number, then $\sum_{d^k|h}\mu(d)=1$, and if h is not a k-th power free number, then $\sum_{d^k|h}\mu(d)=0$, from this properties of the Möbius function and Lemma 2 we have

$$\begin{aligned}&\sum_{n\leq x}\overline{S_2}(n)=\sum_{m^2h\leq x}\sum_{d^2|h}\mu(d)m\\ =\ &\sum_{m\leq\sqrt[4]{x}}m\sum_{h\leq x/m^2}\sum_{d^2|h}\mu(d)+\sum_{h\leq\sqrt{x}}\sum_{m^2\leq\frac{x}{h}}m\sum_{d^2|h}\mu(d)\\ &-\left(\sum_{m\leq\sqrt[4]{x}}m\right)\left(\sum_{h\leq\sqrt{x}}\sum_{d^2|h}\mu(d)\right).\end{aligned}\tag{2}$$

Now we estimate the three terms in the right hand side of formula (2). by Lemma 2 we have

$$\begin{aligned}
& \sum_{m\le\sqrt[4]{x}} m \sum_{h\le x/m^2} \sum_{d^2|h} \mu(d) = \sum_{m\le\sqrt[4]{x}} m \sum_{hd^2\le x/m^2} \mu(d) \\
= & \sum_{m\le\sqrt[4]{x}} m \sum_{d^2\le x/m^2} \mu(d) \sum_{h\le \frac{x}{m^2d^2}} 1 \\
= & \sum_{m\le\sqrt[4]{x}} m \sum_{d^2\le x/m^2} \mu(d) \left[\frac{x}{m^2d^2} + O(1)\right] \\
= & \frac{x}{\zeta(2)} \sum_{m\le\sqrt[4]{x}} \frac{1}{m} + O\left(x^{\frac{3}{4}}\right) \\
= & \frac{x}{\zeta(2)} \left[\frac{1}{4}\ln x + \gamma + O\left(x^{-\frac{1}{4}}\right)\right] + O\left(x^{\frac{3}{4}}\right) \\
= & \frac{3}{\pi^2}\left(\frac{1}{2}\ln x + 2\gamma\right)\cdot x + O\left(x^{\frac{3}{4}}\right), \qquad (3)
\end{aligned}$$

where we have used the identity $\zeta(2) = \frac{\pi^2}{6}$ and $\sum\limits_{n=1}^{\infty} \frac{\mu(n)}{n^2} = \frac{1}{\zeta(2)}$.

$$\begin{aligned}
& \sum_{h\le\sqrt{x}} \sum_{m^2\le\frac{x}{h}} m \sum_{d^2|h} \mu(d) = \sum_{d^2h\le\sqrt{x}} \mu(d) \sum_{m\le\frac{\sqrt{x}}{d\sqrt{h}}} m \\
= & \sum_{d^2h\le\sqrt{x}} \mu(d) \left[\frac{x}{2d^2h} + O\left(\frac{\sqrt{x}}{d\sqrt{h}}\right)\right] \\
= & \frac{x}{2} \sum_{d^2h\le\sqrt{x}} \frac{\mu(d)}{d^2h} + O\left(x^{\frac{3}{4}}\ln x\right) \\
= & \frac{x}{2} \sum_{d^2\le\sqrt{x}} \frac{\mu(d)}{d^2} \sum_{h\le\frac{\sqrt{x}}{d^2}} \frac{1}{h} + O\left(x^{\frac{3}{4}}\ln x\right) \\
= & \frac{x}{2} \sum_{d^2\le\sqrt{x}} \frac{\mu(d)}{d^2} \left[\ln\frac{\sqrt{x}}{d^2} + \gamma + O\left(\frac{d^2}{\sqrt{x}}\right)\right] + O\left(x^{\frac{3}{4}}\ln x\right) \\
= & \frac{x}{2\zeta(2)}\left[\frac{1}{2}\ln x + \gamma\right] - x\sum_{n=1}^{\infty}\frac{\mu(n)\ln n}{n^2} + O\left(x^{\frac{3}{4}}\ln x\right) \\
= & \frac{x}{2\zeta(2)}\left[\frac{1}{2}\ln x + \gamma\right] - \frac{\zeta'(2)}{\zeta^2(2)}\cdot x + O\left(x^{\frac{3}{4}}\ln x\right), \qquad (4)
\end{aligned}$$

where we have used the identity $-\frac{\zeta'(2)}{\zeta^2(2)} = -\sum\limits_{n=1}^{\infty}\frac{\mu(n)\ln n}{n^2}$.

Similarly, we also have the asymptotic formulae

$$\sum_{h\le\sqrt{x}} \sum_{d^2|h} \mu(d) = \sum_{d^2\le\sqrt{x}} \mu(d) \sum_{h\le\frac{\sqrt{x}}{d^2}} 1 = \sum_{d^2\le\sqrt{x}} \mu(d)\left[\frac{\sqrt{x}}{d^2} + O(1)\right] = \frac{\sqrt{x}}{\zeta(2)} + O\left(x^{\frac{1}{4}}\right) \qquad (5)$$

and

$$\sum_{m\le\sqrt[4]{x}} m = \frac{\sqrt{x}}{2} + O\left(x^{\frac{1}{4}}\right). \qquad (6)$$

From (2), (3), (4), (5) and (6) we deduce the asymptotic formula

$$\sum_{n\leq x}\overline{S_2}(n)=\frac{3}{\pi^2}\left(\ln x+3\gamma-1-\frac{12}{\pi^2}\zeta'(2)\right)\cdot x+O\left(x^{\frac{3}{4}}\ln x\right).$$

If $k>2$, then from Lemma 1 and Lemma 3 we have

$$\begin{aligned}
\sum_{n\leq x}\overline{S_k}(n) &= \sum_{m^k h\leq x}\sum_{d^k|h}\mu(d)m=\sum_{m\leq\sqrt[k]{x}} m\sum_{h\leq x/m^k}\sum_{d^k|h}\mu(d)\\
&= \sum_{m\leq\sqrt[k]{x}} m\sum_{hd^k\leq x/m^k}\mu(d)=\sum_{m\leq\sqrt[k]{x}} m\sum_{d^k\leq x/m^k}\mu(d)\sum_{h\leq\frac{x}{m^k d^k}}1\\
&= \sum_{m\leq\sqrt[k]{x}} m\sum_{hd^k\leq x/m^k}\mu(d)=\sum_{m\leq\sqrt[k]{x}} m\sum_{d^k\leq x/m^k}\mu(d)\left[\frac{x}{m^k d^k}+O(1)\right]\\
&= \sum_{m\leq\sqrt[k]{x}} m\sum_{hd^k\leq x/m^k}\mu(d)=\sum_{m\leq\sqrt[k]{x}} m\sum_{d^k\leq x/m^k}\mu(d)\frac{x}{m^k d^k}+O\left(x^{\frac{2}{k}}\right)\\
&= \frac{\zeta(k-1)}{\zeta(k)}x+O\left(x^{\frac{2}{k}}\right). \qquad (7)
\end{aligned}$$

Combining (6) and (7) we may immediately deduce the asymptotic formula

$$\sum_{n\leq x}\overline{S_k}(n)=\begin{cases}\frac{3}{\pi^2}\left(\ln x+3\gamma-1-\frac{12}{\pi^2}\zeta'(2)\right)\cdot x+O\left(x^{\frac{3}{4}}\ln x\right), & \text{if } k=2;\\ \frac{\zeta(k-1)}{\zeta(k)}x+O\left(x^{\frac{2}{k}}\right), & \text{if } k>2.\end{cases}$$

where $\zeta(s)$ is the Riemann zeta-function.

Now we use the analytic method to prove our Theorem. It is clear that $\overline{S_k}(n)$ is a multiplicative function, so for any real number $s>2$, from the Euler product formula we have

$$\begin{aligned}
f(s)=\sum_{n=1}^{\infty}\frac{\overline{S_k}(n)}{n^s} &= \prod_p\left(1+\frac{\overline{S_k}(p)}{p^s}+\frac{\overline{S_k}(p^2)}{p^{2s}}+\cdots+\frac{\overline{S_k}(p^k)}{p^{ks}}+\cdots\right)\\
&= \prod_p\left(1+\frac{1}{p^s}+\frac{1}{p^{2s}}+\cdots+\frac{1}{p^{(k-1)s}}+\frac{p}{p^{ks}}+\frac{p}{p^{(k+1)s}}+\cdots+\frac{p}{p^{(2k-1)s}}+\cdots\right)\\
&= \prod_p\left\{\frac{1-\frac{1}{p^{ks}}}{1-\frac{1}{p^s}}\left(1+\frac{p}{p^{ks}}+\frac{p^2}{p^{2ks}}+\cdots+\right)\right\}\\
&= \prod_p\left\{\frac{1-\frac{1}{p^{ks}}}{1-\frac{1}{p^s}}\left(1+\frac{1}{p^{ks-1}}+\frac{1}{p^{2(ks-1)}}+\cdots\right)\right\}\\
&= \frac{\zeta(s)\zeta(ks-1)}{\zeta(ks)}.
\end{aligned}$$

where $\zeta(s)$ is the Riemann zeta-function.

It is clear that if $k=2$, then function $G(s)=\frac{\zeta(s)\zeta(2s-1)}{\zeta(2s)}\frac{x^s}{s}$ has a pole point at $s=1$ with order 2. The residue of the function $G(s)$ at point $s=1$ is

$$\frac{3}{\pi^2}\left(x\ln x+3\gamma-1-\frac{12}{\pi^2}\zeta'(2)\right).$$

If $k > 2$, then the function $G(s) = \dfrac{\zeta(s)\zeta(ks-1)}{\zeta(ks)}\dfrac{x^s}{s}$ has a simple pole point at $s = 1$ with residue $\frac{\zeta(k-1)}{\zeta(k)}x$. Then by the Perron's formula (see [6]) we have

$$\sum_{n\leq x}\overline{S_k}(n) = \frac{1}{2\pi i}\int_{2-iT}^{2+iT}\frac{\zeta(s)\zeta(ks-1)}{\zeta(ks)}\frac{x^s}{s}ds + O\left(\frac{x^{2+\epsilon}}{T}\right). \tag{9}$$

Moving the integral line in (9) from $s = 2 \pm iT$ to $s = \frac{1}{2} \pm T$, we may immediately deduce the asymptotic formula

$$\sum_{n\leq x}\overline{S_k}(n) = \begin{cases} \dfrac{3}{\pi^2}\left(\ln x + 3\gamma - 1 - \dfrac{12}{\pi^2}\zeta'(2)\right)\cdot x + O\left(x^{\frac{3}{4}+\epsilon}\right), & \text{if } k = 2; \\ \dfrac{\zeta(k-1)}{\zeta(k)}x + O\left(x^{\frac{1}{2}+\epsilon}\right), & \text{if } k > 2. \end{cases}$$

where ϵ denotes any fixed positive number, $\zeta'(2) = -\sum\limits_{n=1}^{\infty}\dfrac{\ln n}{n^2}$.

Now combining two methods we may immediately deduce our Theorem.

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All these papers are original and have been refereed. The themes of these papers range from the mean value or hybrid mean value of Smarandache type functions, the mean value of some famous number theroretic functions acting on the Smarandache sequences, to the convergence property of some infinite series involving the Smarandache type sequences.

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